The Dawn of an Algebraic Era in Discrete Geometry?

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PART I

Seven Wonders of 2010

(in discrete geometry and environs)

The Erdős distinct distances problem

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- known: $g(n) = O(n/\sqrt{\log n})$

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- **Guth and Katz, 2010:** $g(n) = \Omega(n/\log n)$
- Completion of bold ideas of **Elekes** (and Dvir and Sharir and...)

Size of ε-nets for geometric set systems

- $X \subset \mathbb{R}^d$ finite, $\varepsilon > 0$ given
- Wanted: ε-net w.r.t. halfspaces, i.e., $N \subseteq X$ that intersects all *large halfspaces* H = ones with $|H \cap X| \geq \varepsilon |X|$.
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- Based on a general result and known since 1987: $|N| = O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ suffices.
- **Alon 2009:** first superlinear lower bound.
- **Pach and Tardos 2010:** $|N| = \Omega(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ really needed, for $\mathbf{d \geq 4}$.
  - More: ε-nets w.r.t. *axis-parallel rectangles in the plane* ... order $\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}$.
- Lesson: a Ramsey-type theorem can sometimes solve "irregularity of distribution" problem, even quantitatively!

Selection lemma topologically

- P ... a set of n points in $\mathbb{R}^d$
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- **Gromov** 2010: amazing new topological proof, with substantially improved lower bound for c_d.
- **Karasev 2010:** 2-page elementary version.

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- Hirsch: the graph of a d-dimensional convex polytope with n facets has diameter at most $n - d$?!?
- **Santos 2010:** NO, diameter can be at $\geq (1 + \varepsilon)(n - d)$, for some specific small ε_0.
- Upper bound only $n^{O(\log d)}$ [Kalai, Kleitman]

The long randomized simplex

- The simplex method for linear programming: very good in practice, but is it polynomial??
- Klee & Minty NO, exponential for several deterministic pivoting rules.
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- **Friedmann, Dueholm, Zwick 2010:** RANDOM FACET no better than that, and RANDOM EDGE also at least about $e^{n^{1/4}}$.
- PR: also won the Zadeh $1000 bet.
- New technology for building hard instances of linear programs from randomized parity games.

Balanced colorings semidefinitely

- Roth: for an arbitrary red/blue coloring of $\{1, 2, \ldots, n\}$, there is always an arithmetic progression where one color outnumbers the other by at least $cn^{1/4}$.
- One of many problems in discrepancy theory.
- Beck's partial coloring method & some more tricks: a coloring with discrepancy $O(n^{1/4})$ exists.

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- Beck's partial coloring method & some more tricks: a coloring with discrepancy $O(n^{1/4})$ exists.
- **Bansal 2010:** such a coloring can be computed in polynomial time (well, up to some logs).
- Also makes many other results on discrepancy constructive.
- Semidefinite relaxation, rounding by SDP-driven random walk.
- New structural understanding; e.g., discrepancy of a union of two set systems.

Log-concavity for the chromatic polynomial

- A graph G; $p(n) :=$ number of proper colorings of G with n colors.
- Fact: $p(n)$ is a polynomial: the chromatic polynomial.
- $p(x) = \sum_{i=0}^{d} a_i x^i$.

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- $p(x) = \sum_{i=0}^{d} a_i x^i$.
- **Huh 2010:** $(a_0, a_1, \ldots, a_d)$ log-concave, and consequently unimodal.
- Singularities of local analytic functions, mixed multiplicities of ideals,...

PART II

Distinct Distances and Other Algebraic Magic

Erdős' 1946 problems

- $I(m, n)$, maximum possible number of incidences of m points and n lines in the plane;
- maximum possible number of unit distances among n points in the plane;
  - $\approx$ maximum possible number of incidences of n points and n unit circles in the plane
- minimum possible number of distinct distances determined by n points in the plane.

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- in particular, $I(n, n) = \Theta(n^{4/3})$
- for unit distances, the same methods also yield $O(n^{4/3})$
- but lower bound smaller than $n^{1+\delta}$ for every fixed $\delta > 0$.

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- [M.] there exist strictly convex norms admitting only $O(n \log n \log \log n)$ unit distances.

Distinct distances (almost) resolved

- Guth and Katz: $\Omega(n/\log n)$ distinct distances; tight up to $\sqrt{\log n}$.
- Several ideas:
  - Elekes: count isometries mapping some pair of points of P to another;
  - convert the problem to restricted point/line incidences in $\mathbb{R}^3$;
  - Dvir: low-degree polynomial vanishing on the lines;
  - 19th century algebraic geometry (reguli, ruled surfaces, flecnode polynomial,...);
  - polynomial partitions.
- Simplifications & new discoveries surely forthcoming.

Partitioning by polynomials

- $P \subset \mathbb{R}^d$ an n-point set
- $f \in \mathbb{R}[x_1, \ldots, x_d]$ polynomial in d variables;
  $Z = Z(f) \subseteq \mathbb{R}^d$ its zero set
- **Definition:** f is an r-partitioning polynomial for P if no component of $\mathbb{R}^d \setminus Z(f)$ contains more than n/r points of P.

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  - Proof:
    $Z(f_1)$ bisects P into P_1, P_2;
    $Z(f_2)$ bisects P_1, P_2 into P_3, P_4, P_5, P_6;
    $Z(f_3)$ bisects P_3, P_4, P_5, P_6 into $P_7, P_8, \ldots, P_{14}$ (8 sets)
    ... finish when there are more than r sets.
  - $f := f_1 \times f_2 \times \cdots$; count degree.

Partitioning by polynomials

- **Theorem [Guth, Katz]:** Every $P \subset \mathbb{R}^d$ has an r-partitioning polynomial of degree $O(r^{1/d})$.
- Similar to earlier tools like cuttings and simplicial partitions.
- (Currently) less algorithmic, but may be more powerful in some respects ($d \geq 3$).
- Problem (& work in progress): what if many points of P end up in $Z(f)$ (and thus not really partitioned)?

Szemerédi,Trotter: $\mathbf{I(n,n) = O(n^{4/3})}$

- **Proof** [Kaplan,M.,Sharir]; probably observed by others too.
- **(Old) lemma:** $I(P, L) \leq |L| + |P|^2$.

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- Now $P \subset \mathbb{R}^2$ n points; L n lines. Set $r := n^{2/3}$, f is an r-partitioning polynomial, $\deg(f) = O(n^{1/3})$.
  - P_0 points in $Z = Z(f)$;
  - $P_1, P_2, \ldots,$ points in components of $\mathbb{R}^2 \setminus Z$;
  - L_0 lines contained in Z;
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- $|L_0| \le D = O(n^{1/3})$; responsible for $O(n^{4/3})$ incidences.
- $\ell \in L \setminus L_0 \Rightarrow |\ell \cap Z| \le D = O(n^{1/3})$.
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- Remains:
  - $\sum_{i \geq 1} I(P_i, L_i) \overset{lemma}{\leq} \sum_i |L_i| + \sum_i |P_i|^2$
  - $\sum_i |L_i| \leq (D+1)|L| = O(n^{4/3})$
  - $\sum_i |P_i|^2 \leq (\max_i |P_i|) \sum_i |P_i| \leq (n/r) \cdot n = O(n^{4/3})$.

Enriching discrete geometry with algebra

- Sum-product theorems, important mainly in finite fields. Elekes; Bourgain, Katz, and Tao; ...
- Purdy's conjecture and similar. Elekes and Rónyai; Elekes, Simonovits, and Szabó
- Joints of lines in $\mathbb{R}^3$: Guth, Katz; Elekes, Kaplan, Sharir, Shustin; Quillodrán; building on Dvir's trick.
- Unit distances with rational angles $O(n^{1+\delta})$: Schwartz, Solymosi, De Zeeuw.
- Surely I forgot/overlooked some ...

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From: Jiri Matousek <matousek@kam.mff.cuni.cz>
To: Hana Polisenska <poli@kam.mff.cuni.cz>
Subject: prosim objednat knizky

Dobry den,

tady je prosim ten seznam knizek na objednani,
s pozdravem

Jirka Matousek

Ideals, Varieties, and Algorithms: An Introduction to Computational
Algebraic Geometry and Commutative Algebra (Undergraduate Texts in
Mathematics) [Paperback]
David A. Cox (Author), John Little (Author), Donal O'Shea (Author)

Using Algebraic Geometry (Graduate Texts in Mathematics) [Paperback]
David A. Cox John Little (Author), Donal O'Shea (Author)
Publisher: Springer; 2nd edition (March 17, 2005)

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Sheldon Axler (Author)
Publisher: Springer; 2nd edition (July 18, 1997)

Commutative Algebra: with a View Toward Algebraic Geometry (Graduate Texts
in Mathematics) by David Eisenbud
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Publisher: Springer (January 25, 2000)

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