The Dawn of an Algebraic Era in Discrete Geometry?

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Section One, Where the Author Is Browsing the 2010 ArXiv

To me, 2010 looks as *annus mirabilis*, a miraculous year, in several areas of my mathematical interests. Below I list seven highlights and breakthroughs, mostly in discrete geometry, hoping to share some of my wonder and pleasure with the readers.

Of course, hardly any of these great results have come out of the blue: usually the paper I refer to adds the last step to earlier ideas. Since this is an extended abstract (of a nonexistent paper), I will be rather brief, or sometimes completely silent, about the history, with apologies to the unmentioned giants on whose shoulders the authors I do mention have been standing.[1] A careful reader may notice that together with these great results, I will also advertise some smaller results of mine.

- Larry Guth and Nets Hawk Katz [16] completed a bold project of György Elekes (whose previous stage is reported in [10]) and obtained a near-tight bound for the **Erdős distinct distances problem**: they proved that every n points in the plane determine at least $\Omega(n/\log n)$ distinct distances. This almost matches the best known upper bound of $O(n/\sqrt{\log n})$, attained for the $\sqrt{n} \times \sqrt{n}$ grid. Their proof and some related results and methods constitute the main topic of this note, and will be discussed later.

- János Pach and Gábor Tardos [27] found tight lower bounds for the **size of ε-nets** for geometric set systems.[2] It has been known for a long time that for systems such as halfspaces, balls, simplices in $\mathbb{R}^d$, for d fixed, ε-nets of size $O(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$ exist, for every finite set X and every ε. The proof was based on a very general combinatorial property of the considered set system, called *bounded VC-dimension*, and there was hope that simple geometrically defined systems might admit even smaller ε-nets, perhaps of size $O(\frac{1}{\varepsilon})$. Indeed, there were some reasons for optimism, since $O(\frac{1}{\varepsilon})$ was known for halfspaces in $\mathbb{R}^3$, and more recently, Aronov, Ezra, and Sharir [2] proved an $O(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon})$ upper bound for axis-parallel rectangles in $\mathbb{R}^2$. However, Alon [1] established the first superlinear lower bound in a geometric setting (for lines in the plane), and Pach and Tardos got the tight lower bound of $\Omega(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$ for halfspaces in $\mathbb{R}^4$, as well as $\Omega(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon})$ for axis-parallel rectangles in $\mathbb{R}^2$. These results may perhaps not look as significant to others, but for me, they close a long open and tantalizing problem, which for myself I considered almost hopeless. Moreover, I think that the proofs contain (reinforce?) a general lesson: in order to prove an "irregularity" result, in the sense that something cannot be very uniform, it may often be good to strive for a Ramsey-type result, showing that there has to be a completely non-uniform, "monochromatic" spot—in the case of ε-nets, this way even gives a tight quantitative bound!

- Mikhail Gromov [14] invented a new **topological proof of the first selection lemma**. The lemma states that for every n-point set $P \subset \mathbb{R}^d$ there exists a point a (not necessarily in P) contained in at least $c_d\binom{n}{d+1}$ of the d-dimensional simplices with vertices in P, where c_d is a positive constant depending only on d. (There are $\binom{n}{d+1}$ d-simplices spanned by P in total, so a is in a positive fraction of them.) Gromov's proof yields significantly better value of c_d than all previous ones, provides a far-reaching generalization, and hopefully opens a way towards deeper understanding of many related problems. By including Gromov's paper in my 2010 list I am cheating slightly, since a preprint was circulated one or two years earlier. But a completely honest 2010 item was supplied by Karasev [21], who found a greatly simplified and fairly elementary version of the argument. Readers interested in the com-

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[1] I should also add that my selection is entirely personal and subjective, shall not indicate or imply any ranking of the results listed, any judgment of results not mentioned, or any comparison thereof, and shall not encompass any warranty of sound and safe conditions of the cited papers or any liability on my side, in particular, no rights of readers or third parties to be indemnified in case of loss or damage directly or indirectly related to the aforementioned papers. But I do not want this abstract to look like a car rental contract.

[2] Let $\mathcal{F}$ be some system of subsets of $\mathbb{R}^d$, say all closed halfspaces or all axis-parallel boxes, and let X be a finite set in $\mathbb{R}^d$. A subset $N \subseteq X$ is called an *ε-net* for X with respect to $\mathcal{F}$, where $\varepsilon \in [0,1]$ is a given real number, if N intersects every "large" set, i.e., every $F \in \mathcal{F}$ for which $|F \cap X| \geq \varepsilon|X|$.

This is an extended abstract of a presentation given at EuroCG 2011. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear in a conference with formal proceedings and/or in a journal.

binatorial issues involved and in attempts at explaining Gromov's method may also reach for a paper of Wagner and mine [25].

- Francisco Santos [28] disproved the 1957 **Hirsch conjecture**, which states that the graph of a d-dimensional convex polytope with n facets has diameter at most $n-d$. The conjecture was motivated by linear programming, more precisely, by the analysis of simplex-type algorithms. Santos' ingenious examples have diameter $(1+\varepsilon)(n-d)$ for a small positive ε, and the fascinating question of maximum diameter of a d-polytope with n facets has become even more interesting (and the subject of Gil Kalai's polymath project), the best upper bound being $n^{O(\log d)}$ [18].

- Oliver Friedmann, Thomas Dueholm Hansen, and Uri Zwick [13] proved very strong **lower bounds for various randomized simplex algorithms**.

  The simplex method from the late 1940s remains one of the best linear programming algorithms in practice, but a construction known as the *Klee–Minty cube* showed in the 1970s that some of the widely used variants are exponential in the worst case. This started a quest for a polynomial-time version, and on the optimistic side, a subexponential upper bound, of roughly $\exp(O(\sqrt{n}))$, was proved in 1992 for an algorithm known as RANDOM FACET. There was hope that the analysis might be further improved, or that some other randomized variant could be shown to be polynomial.

  Friedmann et al. shattered these great expectations almost completely, by proving an almost matching lower bound for RANDOM FACET, as well as a similar lower bound for another promising-looking algorithm known as RANDOM EDGE (lower bounds of this kind were known before, but only for the performance of these algorithms on certain "generalized linear programs", while the possibility of polynomial bounds for actual linear programs was still open). They use a powerful new way of constructing "difficult" linear programs, based on randomized *parity games*. The potential of this approach apparently has not yet been exhausted.

- Nikhil Bansal [3] found a polynomial-time **algorithm for computing low-discrepancy colorings**,[3] using semidefinite programming and ingenious rounding via a high-dimensional random walk. The algorithm is not an approximation algorithm for discrepancy in general (indeed, approximating discrepancy is pretty much hopeless [5]), but it makes several existential bounds for the discrepancy of certain set systems, such as all arithmetic progressions on $\{1,2,\dots,n\}$, constructive, which has been a major open problem in discrepancy theory. It also has structural consequences; one of the quick spinoffs is a near-tight answer [24] to an old questions of Sós concerning the discrepancy of a union of two set systems on the same ground set (besides Bansal's algorithm, the answer also relies on a beautiful linear-algebraic lower bound for discrepancy of Lovász, Spencer, and Vesztergombi [22]).

- June Huh [17] proved **log-concavity of the sequence of coefficients of the chromatic polynomial**.[4] More precisely, for the chromatic polynomial written as $a_0+a_1x+\dots+a_nx^n$, Huh's result asserts that $a_{i-1}a_{i+1}\le a_i^2$ for every i (and for an arbitrary graph G, of course). This implies that the sequence $(|a_0|,|a_1|,\dots,|a_n|)$ is *unimodal*, a 1968 conjecture of Read.

  The proof relies on connections of the problem to singularities of local analytic functions and ultimately to mixed multiplicities of certain ideals. This result does not entirely fit my list since it cannot be passed for discrete geometry even with considerable indulgence, but it looks beautiful and it does rest on geometric ideas. I do not understand much of it, but perhaps some day I will have enough time and energy to learn the necessary background or someone will explain it to me—at least it does not look as intimidating as some other papers.

Section Two, On Distinct Distances and Other Algebraic Magic

The following three problems were raised by Erdős [12] in 1946:

- Estimate the maximum possible number of **incidences** between a set P of m points and a set L of n lines in the plane (where an incidence is a pair (p,ℓ) with $p\in P$, $\ell\in L$, and $p\in\ell$).

- Estimate the maximum number of **unit distances** among n points in the plane. This can also be reformulated using incidences; up to a multiplicative factor of at most 2, one wants to estimate the maximum number of incidences between n unit circles and n points in the plane.

[3] The input to the algorithm is a set system $\mathcal{F}$ on a finite set X, and the output is a coloring χ of X by +1s and −1s. The *discrepancy* $\mathrm{disc}(\chi,\mathcal{F})$ of χ is the maximum, over all sets $F\in\mathcal{F}$, of the "imbalance" of F, i.e., of $\left|\sum_{x\in F}\chi(x)\right|$. The discrepancy of $\mathcal{F}$ is $\min_\chi \mathrm{disc}(\chi,\mathcal{F})$, where the minimum is over all possible ±1 colorings of X.

[4] The *chromatic polynomial* of a graph G is a polynomial whose value at a natural number k equals the number of proper coloring of G with k colors. It can be shown that such a polynomial indeed exists, and is determined uniquely, for every G.

- Estimate the minimum number of **distinct distances** determined by n points in the plane.

The first two of these problems, incidences and unit distances, look quite similar at first sight. For point-line incidences the order of magnitude was determined precisely by Szemerédi and Trotter [31] in 1983; in particular, for n points and n lines the bound is of order $n^{4/3}$. Two other, much simpler proofs were discovered later, by Clarkson et al. [6] and by Székely [30]. These proofs are "combinatorial", in the sense that they do not use the straightness of the lines (and thus they also work for incidences of points with *pseudolines*, i.e., with systems of curves satisfying certain simple combinatorial axioms).

For unit distances, or incidences of unit circles with points, a modification of these proofs also yields an $O(n^{4/3})$ upper bound, but here a matching lower bound is lacking—the best known example, a suitable grid, yields only a near-linear bound, smaller than $n^{1+\delta}$ for every fixed $\delta > 0$. Erdős conjectured this lower bound to be essentially optimal.

Many ingenious techniques have been developed for bounding the number of incidences in various settings; see, e.g., a recent survey by Pach and Sharir [26]. However, none of them seems capable of breaking the $n^{4/3}$ barrier. Moreover, the unit distance problem can be considered not only for the Euclidean norm, but also for other norms in the plane, and Valtr [32] constructed a norm for which n-point sets actually exist with $\Omega(n^{4/3})$ unit distances. This indicates that limits of combinatorial approaches to the unit distance problem, powerful as they are, have been reached.

It seems that for further progress on unit distances (in the upper bound, which should be the way to go according to Erdős and general belief) one has to use some algebraic properties of the circle.

Indeed, while the unit distances problem still stands, the distinct distances problem was essentially settled by Guth and Katz, as was already mentioned—and their proof combines ingenious algebraic arguments (using tools from algebraic geometry, mostly dating back to the 19th century) with geometric, or perhaps topological, considerations similar to those appearing in earlier higher-dimensional incidence proofs.

The Guth–Katz proof is somewhat complicated (and I am sure simpler variants will be found sooner or later), and I will not attempt to even sketch it here. I will just mention two of its main ingredients.

One of them is a simple but surprisingly powerful trick invented by Dvir [7], which can be illustrated on another remarkable recent achievement in incidence problems, the *joints problem* (solved in another paper of Guth and Katz [15], with further simplifications and extensions by Elekes, Kaplan, Sharir, Shustin, and Quillodrán; the following outline mostly follows [19]). We consider a set L of n lines in $\mathbb{R}^3$, and call a point $a \in \mathbb{R}^3$ a *joint* if there are at least three lines of L, not all coplanar, passing through a. The question is, what is the maximum possible number of joints for n lines, and the answer is $O(n^{3/2})$ (with a matching lower bound example provided by a suitable grid of lines). The proof goes as follows.

For contradiction, we suppose that a set L of n lines has at least $Cn^{3/2}$ joints, with C very large (and n even much larger). Let J be the set of all joints of L, and let $m := |J|$.

We first need to prune L and J so that all of the remaining lines contain many joints. By a simple pruning procedure (repeatedly discarding lines with a small number of joints and all the joints incident to them) we can select $L' \subseteq L$ and $J' \subseteq J$ so that every $\ell \in L'$ contains at least $k := m/2n$ points of J', and each point of J' is a joint of the lines in L'.

Next, a simple argument shows that there exists a nonzero polynomial in three variables that vanishes on all points of J' and has degree at most $4m^{1/3}$ (the condition of vanishing on J' is expressed by a system of homogeneous linear equations with the coefficients of the desired polynomial as unknowns, and when the number of unknowns exceeds the number of equations, a nonzero solution exists). Among all nonzero polynomials vanishing on J' we choose one of the smallest possible degree and call it f; we thus have $D := \deg(f) \leq 4m^{1/3}$.

Let us consider the restriction f_ℓ of f to a line $\ell \in L'$. It vanishes at each of the at least k points of J' incident to ℓ. Since f_ℓ can be regarded as a univariate polynomial of degree at most D, either it is 0 everywhere on ℓ, or it has at most D zeros there. But for large C we have $D < k$, and so we get that $f_\ell \equiv 0$. Thus f vanishes on all lines of L'.

Now we let $\vec{g} := \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ be the gradient of f (this is a polynomial mapping $\mathbb{R}^3 \to \mathbb{R}^3$). Since f vanishes along every $\ell \in L'$, the projection of $\vec{g}(p)$ onto ℓ also vanishes for all $p \in \ell$. In other words, $\vec{g}(p)$ is perpendicular to ℓ. In particular, if we look at a joint $a \in J'$, we obtain that $\vec{g}(a)$ is perpendicular to three non-coplanar lines, and hence $\vec{g}(a) = (0, 0, 0)$.

Since taking a partial derivative decreases the degree of a polynomial, each of the three components of $\vec{g}$ is a polynomial of degree *smaller* than D, and we have just proved that it vanishes on J'. Since f was the smallest-degree nonzero polynomial vanishing on J', $\vec{g}$ must be identically 0. This means than f is constant, and this is a contradiction (since we assumed it to be nonzero and to vanish on J'), which concludes the proof.

This was Dvir's trick in action, and now, what *is* the trick? An idea turned into a recipe often loses much of its usefulness, but a vague general formulation might perhaps be this: one constructs a lowdegree polynomial vanishing on the considered points, or other "low-dimensional" objects, infers that it has to vanish on some other objects of higher dimension as well, and derives a global conclusion about the polynomial, such as vanishing of the gradient.

Another ingredient of the distinct distances proof, which was invented by Guth and Katz for this purpose, are *space partitions using polynomials*. Given a finite set $P \subset \mathbb{R}^d$ and a parameter r, $1 < r \leq |P|$, they apply the *polynomial ham-sandwich theorem* of Stone and Tukey to obtain a partition of $\mathbb{R}^d$. Namely, they construct a nonzero polynomial f so that no connected component of $\mathbb{R}^d \setminus Z(f)$ contains more than $|P|/r$ points of P (where $Z(f)$ is the zero set of f). A key feature of the construction is that $\deg(f)$ need not be too high, only $O(r^{1/d})$, and thus the interaction of other objects, such as lines or hyperplanes, with $Z(f)$ is under control in some sense. This method provides an alternative to previous space partitioning techniques, such as *cuttings* or *simplicial partitions*, and in some respects it is apparently more powerful than these earlier tools.

A simple example of using these polynomial partitions is a new simple proof of the Szemerédi–Trotter upper bound on point-line incidences. Together with some other applications, we have described this in detail in a recent preprint with Kaplan and Sharir [20], and so it will not be discussed here.

There are several other encouraging examples of "enriching discrete geometry with algebra" besides those mentioned above, a major share of them due to Elekes and his co-authors.

One of the major directions are *sum-product theorems*, originally a number-theoretic issue. The earlier stages, dealing with real numbers, are nicely surveyed in Elekes [8]. Then, after Bourgain, Katz, and Tao [4] obtained a sum-product theorem in finite fields, there has been a great surge of activity, and the sum-product business has been extended far beyond the borders of discrete geometry (and far beyond the intended scope of the present note). It has found many diverse applications, including in number theory, group theory, and the explicit constructions of extractors and Ramsey graphs. An accessible initial picture and references can be gained, e.g., from a talk of Wigderson as recorded in the lecture notes at `http://www.math.cmu.edu/~af1p/Teaching/AdditiveCombinatorics/allnotes.pdf`.

Another interesting (and related) direction can be represented, e.g., by the papers of Elekes and Rónyai [9] and of Elekes, Simonovits, and Szabó [11]. I will mention only a rather concrete consequence of the results of [9], known as *Purdy's conjecture*. Let C be a constant, let a and b be lines in the plane, let $p_1, \dots, p_n$ be distinct points on a, and let $q_1, \dots, q_n$ be distinct points on b. Purdy's conjecture, now a theorem, states that if the set of the Euclidean distances $\{\|p_i - q_j\| : i, j = 1, 2, \dots, n\}$ has at most Cn distinct elements, and if n is sufficiently large, then a and b must be either parallel or perpendicular. Actually, it seems that it should be sufficient to assume at most $n^{2-\varepsilon}$ distinct distances, for some fixed $\varepsilon > 0$, and the conclusion should still hold, but this is wide open at present.

The last two "algebraic" contributions mentioned here concern the unit distances problem, or rather, they deal with variants of it. Schwartz et al. [29] proved that if we count only unit distances with *rational angles* (i.e., attained for pairs of points p, q such that the angle of the line pq with the x-axis is a rational multiple of π), then the number of these special unit distances is bounded by $O(n^{1+\varepsilon})$, for every fixed $\varepsilon > 0$. In [23], I proved that there *exist* norms in the plane for which the number of unit distances obeys the upper bound $O(n \log n \log\log n)$ (while *every* norm admits sets with $\Omega(n \log n)$ unit distances, and, according to Valtr [32], *some* norms allow for as many as $\Omega(n^{4/3})$ unit distances). The proof combines a linear-algebraic argument, a graph-theoretic lemma we proved earlier for another purpose with Přívětivý and Škovroň, and a Baire category argument.

The above short survey certainly does not exhaust all significant examples of algebraic methods used in discrete geometry. Moreover, I have no doubts that, while I am writing this, bright mathematicians are working hard on extending the Guth–Katz breakthrough and on other amazing algebraic methods and applications.

So, does this signify the beginning of an algebraic era in discrete geometry? My list at the beginning of this note shows a more balanced picture, with topology, probability, optimization, and ingenuity in a friendly competition with algebra, but it indicates that the best results have usually been achieved not only by cleverness and new ideas, but also by applying more and more advanced tools from various branches of mathematics. So I am not sure about an algebraic era, but, just in case, recently I have ordered five textbooks on algebraic geometry and such things for my PhD students and myself.

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