Construction of Common Unfolding of a Regular Tetrahedron and a Cube

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Abstract

A procedure that produces a common unfolding of a regular tetrahedron and a cube is given. It is adaptable to the length of an edge. If we allow a small error of the length of an edge of the tetrahedron, the procedure certainly halts and generates a common unfolding of a cube and an almost regular tetramonohedron. If we wish to generate the common unfolding of them with accuracy, we conjecture that the procedure does not halt and we obtain the common unfolding in the limit as a set of infinitely many points. The procedure has a potential to design a fractal structure given in a continued fraction form.

1 Introduction

Recently, several polygons that can fold to two different polyhedra have been developed (Table 1). Such a polygon is called a *common unfolding* of the polyhedra[1]. Observing these impressive unfoldings, it is natural to ask that whether there is a common unfolding of two (or more) different Platonic solids. This question has arisen several times independently, and it is still open (see [3, Section 25.8.3]).

In this paper, we give a procedure that generates a common unfolding of a regular tetrahedron and a cube[2]. The procedure is adaptable to the length of an edge of the tetrahedron. More precisely, the procedure generates the common unfolding of a cube and an almost regular tetramonohedron. If we wish to generate the common unfolding of a cube and a regular tetrahedron, the procedure produces a set of an infinite number of points, and we obtain the common unfolding in the limit. In a sense, this procedure gives an affirmative answer to the open problem; there exists a common unfolding of two Platonic solids (if our conjecture based on experiments is true). When we admit some error, say $\epsilon > 0$, then the procedure always halts and it certainly generates a common unfolding of a cube and an almost regular tetramonohedron whose edge lengths are within the interval $[\ell-\epsilon, \ell+\epsilon]$, where $\ell = \sqrt{2\sqrt{3}}$ is the length of an edge of the regular tetrahedron of surface area 6. Experimentally, we obtain a common unfolding of a cube and an almost regular tetramonohedron with $\epsilon < 2.89200 \times 10^{-1796}$.

Although we obtain an unfolding close to the ideal one, the connectivity of the unfolding generated by the procedure is not guaranteed in general. From the experimental results, we conjecture some useful properties of the unfolding. Based on it, we can take arbitrarily small $\epsilon > 0$ to obtain such an unfolding of a cube and an almost regular tetramonohedron.

The behavior of the procedure seems to rely on the continued fraction form of the length of an edge. On the other hand, it is well known that some real numbers have simple (but infinite) continued fraction forms. The conjecture would imply that our procedure is also useful to generate so called fractal curves based on these forms; see Figures 6 and 7 in Appendix.

2 Preliminaries

We first show some basic results about unfolding of a convex polyhedron.

Lemma 1 ([3, Sec. 22.1.3]) *All vertices of a polyhedron X are on the edges of any unfolding of X.*

Let P be a polygon on the plane, and R be a set of three points (called *rotation centers*) on the boundary of P. Then P has a *tiling* called symmetry group *p2* if P fills the plane by the repetition of 2-fold rotations around the points in R. The filling should contain no gaps nor overlaps. The rotation defines an equivalence relation on the points in the plane. Two points p_1 and p_2 are mutually equivalent if p_1 can be moved to p_2 by the 2-fold rotations. More on the properties of p2 tiling can be seen, e.g., in [5]. Based on the notion of p2 tiling, any unfolding of a tetramonohedron[3] can be characterized as follows:

Theorem 2 ([1, 2]) *P is an unfolding of a tetramonohedron if and only if (1) P has a p2 tiling, (2) four of the rotation centers consist in the triangular lattice formed by the triangle faces of the tetramonohedron, (3) the four rotation centers are the lattice points, and (4) no two of the four rotation centers belong to the same equivalent class on the tiling.*

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[1] Note that an edge of an unfolding can pass through a flat face of the polyhedra. See an unfolding of a cube in Figure 1.

[2] In this paper, a *cube* always means a unit cube that is a box of size $1 \times 1 \times 1$.

[3] A *tetramonohedron* is a tetrahedron that consists of four congruent triangles.

A regular octahedron and a tetramonohedron	[3, Figure 25.50]
A regular tetrahedron and a box of size $1 \times 1 \times \sqrt{3} - 1/2 = 1.232$	[3, Figure 25.51]
A cube and a tetramonohedron of size $1 : \sqrt{34}/6 : \sqrt{34}/6 = 0.9718$	[6]
A regular octahedron and a tetramonohedron of size $1.0072 : 0.9965 : 0.9965$	[6]
A regular icosahedron and a tetramonohedron of size $1 : 1.145 : 1.25$	[4]

Table 1: Known common unfoldings of a regular polyhedron and another (nonregular) polyhedron.

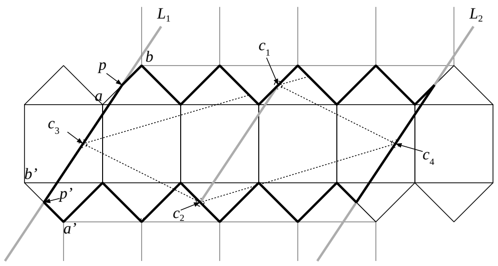

Figure 1: An initial unfolding P_1 of a cube.

We start from an unfolding P_1 of a cube in Figure 1. (The points c_1, c_2, p, p' are the centers of corresponding edges.) The thick lines in the figure gives an unfolding of a cube. Also, P_1 satisfies the conditions of Theorem 2 with four points c_1, c_2, c_3, c_4; that is, P_1 is an unfolding of a tetramonohedron. We here note that c_1c_2 is in parallel to the lines L_1 and L_2, and the tiling is split by lines L_1 and L_2 into "parallel ribbons." Thus we obtain infinitely many tilings by shifting these ribbons into any position along the lines. This fact implies that we can move the rotation centers c_3 and c_4 to *any* positions along L_1 and L_2, respectively, as long as $|c_1c_3| = |c_2c_4|$ (and hence $|c_1c_4| = |c_2c_3|$) without changing the surface area of the tetramonohedron. That is, P_1 is a common unfolding of a cube and an infinitely many tetramonohedra. (In the context of [3, Section 25], L_1 and L_2 make a *rolling belt* that is zipped to the edge c_3c_4.) Here we have $|c_1c_2| = \sqrt{13}/2 = 1.80278$, and each area of four congruent triangles is 3/2. Thus taking $|c_3c_1| = |c_3c_2| = |c_4c_1| = |c_4c_2|$ in Figure 1, we have the following lemma:

Lemma 3 *There exists a common unfolding of a cube and a tetramonohedron with edge lengths* $\sqrt{13}/2 : \sqrt{745/208} : \sqrt{745/208} = 1.80278 : 1.89255 : 1.89255$.

The tetramonohedron in Lemma 3 is close to a regular tetrahedron. Our goal is to modify the edge lengths to the equal length $\sqrt{2\sqrt{3}} = 1.86121$.

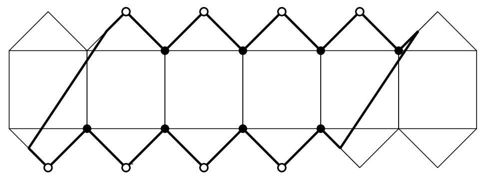

Figure 2: Fixed points on an unfolding.

3 Procedure for a common unfolding

Our procedure transforms P_1. More precisely, it moves c_1 to the right and c_2 to the left, respectively. Through these transformations, we keep two invariants for the polygon P_1 that P_1 is an unfolding of a cube, and P_1 is an unfolding of a tetramonohedron with $|c_1c_3| = |c_1c_4| = |c_2c_3| = |c_2c_4|$. Hence, when $|c_1c_2|$ becomes $\sqrt{2\sqrt{3}}$, we obtain a common unfolding of a cube and a regular tetrahedron. When we move c_1 and c_2, a series of discrete processes occurs. It will not terminate if we try to change from $|c_1c_2| = \sqrt{13}/2$ to $|c_1c_2| = \sqrt{2\sqrt{3}}$ as shown later. Thus we have two choices; the common unfolding of a cube and the regular tetrahedron in the limit as the set of infinitely many points, or a common unfolding of a cube and an *almost regular tetramonohedron* for any given error $\epsilon > 0$. Here we define the *almost regular tetramonohedron* with error $\epsilon > 0$ by a tetramonohedron with $|c_1c_2| \in [\sqrt{2\sqrt{3}} - \epsilon, \sqrt{2\sqrt{3}} + \epsilon]$.

Now we show how to stretch the distance between c_1 and c_2 and change $|c_1c_2|$ from $\sqrt{13}/2 = 1.80278$ to $\sqrt{2\sqrt{3}} = 1.86121$. Intuitively, we will slightly move the points c_1 and c_2 horizontally farther.

We here observe that, white circles in Figure 2 come to two "center points" in the top and bottom squares in the cube. If we remove these points, the squares have holes. On the other hand, if the unfolding contains these points inside, the squares have overlaps. Hence these points should be on the edge of P_1. Moreover, by Lemma 1, the vertices of the cube (black points in Figure 2) also should be on an edge of P_1. We call these immovable points *fixed points* of the unfolding.

Here we focus on the top edges of P_1. There are eight fixed points, and we have a rotation center c_1

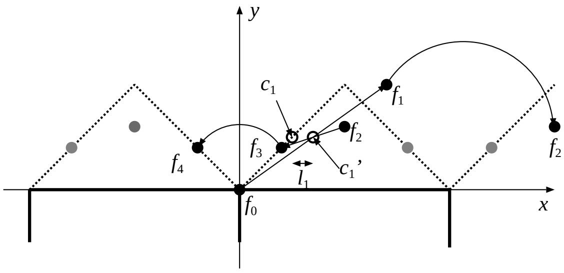

Figure 3: The construction of the points on edges of the unfolding of a cube and a tetramonohedron.

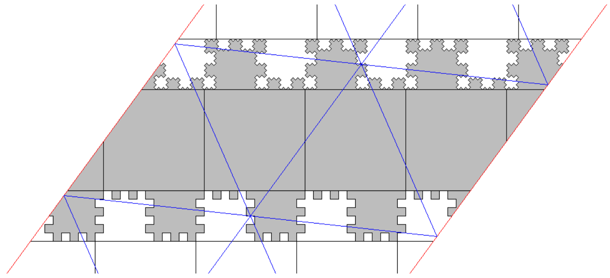

Figure 4: An example of an unfolding of a cube and a tetramonohedron.

on it. To describe exactly, we define xy-coordinate on the edges; let one of the fixed points, or the vertex of the (unit) cube closest to c_1 be $f_0 = (0, 0)$ (Figure 3). Then the initial position of c_1 is $(1/4, 1/4)$. Now, we assume that the rotation center $c_1 = (1/4, 1/4)$ is moved to $c'_1 = (1/4+\ell_1, 1/4)$ for small ℓ_1. Since f_0 is a fixed point and c'_1 should be the rotation center, $f_1 = (1/2 + 2\ell_1, 1/2)$ has to be a point on the edge of the unfolding. Moreover, when this unfolding folds up to a cube, f_1 will be put on the point $f_2 = (1+1/2, 1/2-2\ell_1)$. Hence f_2 is also a point on the edge of the unfolding[4]. Since they form a symmetrical unfolding, when the point $f_2 = (1+1/2, 1/2-2\ell_1)$ is placed, the equivalent point $(1/2, 1/2 - 2\ell_1)$ is also placed. That is, the equivalent relation $(x, y) \equiv (x - 1, y)$ is always applied. We can repeat this process and obtain a set of points which should be on edges of the unfolding with the new rotation center $c'_1 = (1/4 + \ell_1, 1/4)$. In other words, we have a set of points that should be on edges of the unfolding with respect to the shift value ℓ_1. For this procedure, we have the following lemma:

Lemma 4 *The procedure of mapping the fixed points halts if and only if ℓ_1 is a rational number.*

Proof. If $\ell_1 = \frac{p}{q}$ with $0 < p < q$ for some positive integers p and q, all the mapped points from the fixed points are onto a lattice of size $O(pq)$. Hence the procedure can be terminated when the mapping visits some point again. On the other hand, if ℓ_1 is not a rational number, the same coordinate never appears and the procedure will not terminate. □

Using the procedure, we can slightly move c_1 and c_2 horizontally, and stretch $|c_1c_2|$ to the desired length. We now combine the construction in Figure 1; we also tilt the lines L_1 and L_2 in parallel with c_1c_2 consistent to the surface area, and obtain the desired unfolding. For example, Figure 4 is a common unfolding of a cube and a tetramonohedron such that $\ell_1 = 4/21$ and $\ell_2 = 5/24$, where ℓ_2 is the distance that c_2 is moved to the left.

In general, this procedure does not always generates a connected unfolding. More precisely, two lines L_1 and L_2 may cut the unfolding into disconnected pieces. To guarantee that these lines does not divide the unfolding, we have to investigate the generated edges by the procedure. We experimentally generated many unfoldings, and obtain the following observation, but we have no proof and no formal characterization of this sets of points up to now:

Observation 1 *We let $\ell_1 = (1 - \phi_1)/4$ and $\ell_2 = (1 - \phi_2)/4$, where ϕ_1 and ϕ_2 are given in continued fraction forms[5] $\phi_1 = \frac{1}{a_1\pm}\frac{1}{a_2\pm}\frac{1}{a_3\pm}\cdots\frac{1}{a_k}$, and $\phi_2 = \frac{1}{b_1\pm}\frac{1}{b_2\pm}\frac{1}{b_3\pm}\cdots\frac{1}{b_h}$. Then, each upper edge of the unfolding is given by the waves recursively defined by a_i as follows. We first replace a line segment of the original P_1 by a curve with a_1 "waves;" it is a triangular wave or a square wave depending on the parity of a_1. Then, each edge is again replaced by the waves decided by a_2, and so on. The signal determines the first direction of the wave.*

For example, in Figure 4, $\ell_2 = (1 - 1/6)/4 = 5/24$, or $\phi_2 = 1/6$. Hence the lower edges consist of a square wave of 6 peeks. On the other hand, since $\phi_1 = 5/21 = 1/(4 + 1/5)$, each upper edge consists of a square wave of four peeks of which each edge of the square consists of a triangular wave of five peeks.

We conjecture that the observation certainly holds, but no proof and no exact characterization of the "wave" are given. However, we can construct an accurate connected unfolding based on the observation. Precisely, we obtain several values by a brute force algorithm that alternately chooses $a_1, b_1, a_2, b_2, \ldots$ to

[4] Precisely, before computing this "rotation" of the point f_1 to f_2, we have to determine if f_1 is on the left side or on the right side of the unfolding. Depending on the side, we have to choose the direction of the rotation from clockwise and counterclockwise. When the point f_i is just on the boundary, or when $f_i = (0, j)$ or $f_i = (0.5, j)$ with $j > 0$, we need some "clue;" to decide that, we also maintain which side is "inside" of the unfolding for each point f_i. But the details are omitted in this draft.

[5] This "continued fraction form" is slightly different from the standard one. In the standard form, $a_i \geq 1$ and all signs are "+." In our continued fraction form, we use "−" and avoid the case $a_i = 1$ for $i > 1$. The details are omitted in this draft.

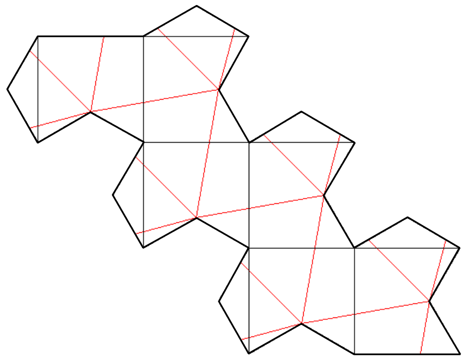

Figure 5: A common unfolding of a cube and a (nonregular) octahedron.

be closer $\ell_1 + \ell_2 = \sqrt{2\sqrt{3} - 9/4} - 1$: $a_1 = 4, b_1 = 6, a_2 = 6, b_2 = -34, a_3 = -42, b_3 = -14, a_4 = -116, b_4 = -2146, a_5 = 4010, b_5 = -3316, a_6 = -4958, b_6 = 8684, a_7 = -7820, b_7 = 7082, a_8 = 2668, b_8 = -3684, a_9 = 4564, b_9 = 1662, a_{10} = 560, b_{10} = -158, \ldots$. We have obtained a connected unfolding for each i. When we use up to a_{10} and b_{10}, we obtain $\epsilon < 4.63451 \times 10^{-56}$, and the values up to a_{50} and b_{50} give us $\epsilon < 2.89200 \times 10^{-1796}$:

Theorem 5 *There exists a common unfolding of a cube and an almost regular tetramonohedron with an error* $\epsilon < 2.89200 \times 10^{-1796}$.

We conjecture that this process can be repeated in arbitrary large, and we obtain a connected common unfolding of a cube and a regular tetrahedron in the limit:

Conjecture 1 *There exists a series of points that converges to a connected common unfolding of a cube and a regular tetrahedron in the limit.*

4 Concluding Remarks

We give a procedure that generates a common unfolding of a cube and a regular tetrahedron. But this is not completed: the procedure does not halt, and it might produce a disconnected unfolding. The proof of the observation, or the characterization of the curve generated by the procedure is a future work. We also conjecture that a similar construction of a common unfolding of a regular octahedron and a tetramonohedron works.

It is a challenging problem to try to the other pair of Platonic solids rather than a regular tetrahedron; in the case, we cannot use tiling as a tool any more. However, recently, the first author finds a common unfolding of a cube and a (nonregular) octahedron (Figure 5; note that this octahedron consists of two regular triangles of edge length $\sqrt{2}$, and six triangles of edge lengths $\sqrt{2}$, $\sqrt{24 - 6\sqrt{3}}/3$, $\sqrt{24 - 6\sqrt{3}}/3$). Hence it is not so easy to give some negative results, and we might have a common unfolding of the other solids.

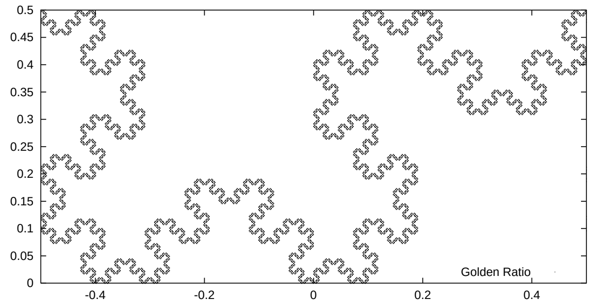

Figure 6: The set of the first 5000 points given by the golden ratio $\phi = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$.

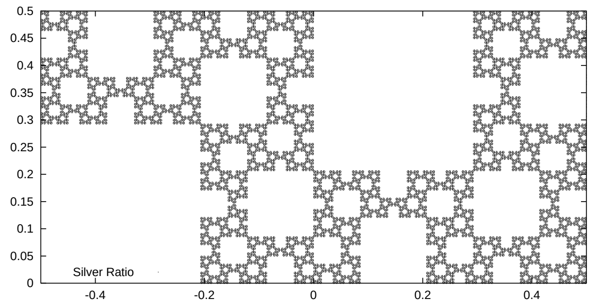

Figure 7: The set of the first 10000 points given by the silver ratio $\phi = \sqrt{2} - 1 = \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots$.

Appendix

Based on the observation, we can design and generate some fractal pattern for a certain real number ϕ. Some experimental results are given in Figures 6 and 7.

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