Long Monotone Paths in Convex Subdivisions

Günter Rote*

Abstract

Consider a connected subdivision of the plane into n convex regions where every vertex has degree at most d. Then, for every vertex there is a path with at least $\Omega(\log_d n)$ edges through this vertex that is monotone in some direction. This bound is best possible.

1 Introduction

Definition 1 *A* direction *is given by a unit vector e. A path $x(t)$, $t \in I$ for some interval I, is* monotone in direction e *if the inner product $x(t) \cdot e$ is a strictly increasing function of t. A path is* weakly monotone in direction e *if the inner product is a nondecreasing function of t.*

We say that a path is (weakly) monotone *if it is (weakly) monotone in some direction.*

We are looking for long monotone paths in planar subdivisions. A subdivision of the plane into n convex regions has $O(n)$ vertices and $O(n)$ straight edges, including some infinite rays. When we speak about monotone paths, we mean paths along the vertices and edges of this graph. To exclude some trivial cases, we assume that the edges form a connected graph (and we say that the subdivision is *connected*).

Theorem 1 *Let P be a connected subdivision of the plane into n convex faces in which every vertex has degree at most d. Then, for every vertex v, there is a weakly monotone path with at least $\Omega(\log_d n)$ edges starting at v.*

Related Results. If P is the vertical projection of a piecewise linear convex terrain $\hat{P}$, one can apply a polarity transform to this terrain with respect to the paraboloid $z = x^2 + y^2$, yielding another piecewise linear convex terrain $\hat{P}^*$. The projection P^* of this terrain is a *reciprocal diagram* of P (cf. [1]): its graph is dual to the graph of P, in the sense that vertices of P^* correspond to faces of P and vice versa. Each edge in P^* has a corresponding edge in P, and moreover, these two edges are perpendicular. (This last property distinguishes a reciprocal diagram from a general drawing of the dual graph of P.)

*Institut für Informatik, Freie Universität Berlin, Takustraße 9, 14195 Berlin, Germany, rote@inf.fu-berlin.de

A monotone path in P becomes a *monotone face sequence* in P^*: in such a sequence, one can go from a face A to an adjacent face B whenever there is a ray in the specified direction e which crosses the common edge of A and B in the direction from A to B.

Monotone face sequences were studied by Chazelle, Edelsbrunner and Guibas [2]. They established the following bound:

Theorem 2 *Let Q be a subdivision of the plane into n convex faces in which every face is adjacent to at most d neighboring faces. Then, there is a monotone face sequence of length at least*

$$\Omega(\log_d n + \log n / \log\log n).$$

Chazelle et al. showed that a monotone sequence of this length can even be achieved by the faces intersected by a line. Moreover, they showed that the bound is tight, by giving families of subdivisions that have no long face sequences.

Thus, for subdivisions P that are projections of a convex terrain, the question about long monotone paths is completely answered the applying above theorem for $Q = P^*$.

However, for general subdivisions, the problems are not directly related, and in fact, the answers are different: We will see in Section 3 that one cannot add a term $\Omega(\log n / \log\log n)$ (or any other term that grows to infinity) to the bound of Theorem 1.

Motivation. We were led to the question of Theorem 2 by the complexity analysis of an algorithm for partial matching between two finite planar point sets under translations [4]. There, we could show that a certain subdivision Q contained no monotone face sequence longer than some bound X which is polynomial in the input parameters. If Theorem 2 were true with some stronger lower bound of the form $\Omega(n^\alpha)$ for $\alpha > 0$, this would have implied a polynomial bound on the number of faces of Q.

2 Proof of Theorem 1

A basic observation for convex subdivisions is that for any vertex v and any given generic direction e, there is always an outgoing edge from v that is weakly monotone in direction e. Theorem 1 follows quite straightforwardly from the following lemma:

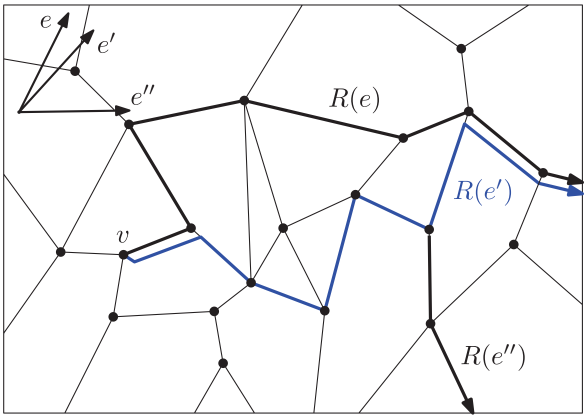

Figure 1: The rightmost path $R(e)$ starting from vertex v in direction e

Lemma 3 *Let v be a vertex in a connected convex planar subdivision P. Then there is a spanning tree rooted a v and containing all infinite rays, such that all paths starting at v are weakly monotone.*

A wall of bricks shows that the lemma does not hold with (strictly) monotone paths. However, if there are no angles of 180°, the statement extends to strictly monotone paths.

Proof. For a generic direction e we can define the rightmost path $R(e)$ starting at v as follows, see Figure 1: We start at v and always follow the rightmost outgoing edge that is weakly monotone in direction e until we arrive at an unbounded ray.

Now we start rotating e clockwise. At some direction e', $R(e')$ will be different from $R(e)$. $R(e)$ is still weakly monotone in direction e'. Now, any vertex u (and any edge) in the region between $R(e)$ and $R(e')$ can be reached by a monotone path in direction e': We simply start at u and go monotonically in the direction opposite to e' until we hit $R(e)$ or $R(e')$. From there, we follow $R(e)$ or $R(e')$ to v. In this way, we can form a spanning tree of all vertices and all infinite rays between $R(e)$ and $R(e')$ with the desired properties.

Continuing the rotation in this way, we eventually reach all vertices and all infinite rays. □

3 Upper Bounds

If the maximum degree d is $\Omega(n)$, Theorem 1 gives only a trivial statement. The example of Figure 2 shows that, indeed, there is no non-constant lower bound on the length of monotone paths in this case, even for a triangulation.

The graph has $2k$ vertices arranged on two concentric rings around a central vertex and connected in a zigzag manner. Each vertex is connected to the center and has also an infinite ray outwards. The radius of the outer ring is (at least) twice as large as the radius of the inner ring. If we disregard the central vertex and the infinite edges, the zigzag remains. One can easily see that a monotone path can use at most three successive edges of the zigzag. Since a path can go through the center at most once and can use at most two infinite rays, a constant upper bound follows. In fact, it is easy to show that a monotone path can have at most eight edges.

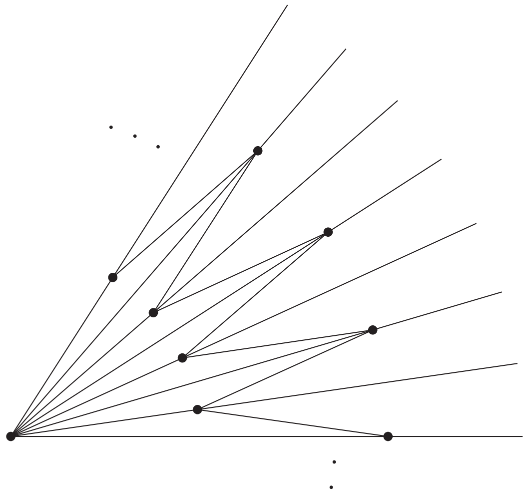

Figure 2: A subdivision with only constant-size monotone paths

4 Open Question

The example of Figure 2 has $\Omega(n)$ infinite rays. Are there similar examples with constantly many rays?

Acknowledgments

This work was done while I was a guest of CIB (Centre Interfacultaire Bernoulli) in Lausanne. I gratefully acknowledge the support of the CIB and the NSF (National Science Foundation). I thank János Pach for insistingly asking the question to which Theorem 1 gives the answer. I thank the organizers for the tasty chocolate.

References

[1] F. Aurenhammer. A criterion for the affine equality of cell complexes in R^d and convex polyhedra in R^{d+1}. *Discrete Comput. Geom.*, 2:49–64, 1987.

[2] B. Chazelle, H. Edelsbrunner, and L. J. Guibas. The complexity of cutting complexes. *Discrete Comput. Geom.*, 4:139–181, 1989.

[3] D. Grace. All theories proven with one graph. *The Journal of Irreproducible Results*, 50(1):1, 2006.

[4] G. Rote. Partial least-squares point matching under translations. In J. Vahrenhold, editor, *26th European Workshop on Computational Geometry (EuroCG'10)*, pages 249–251, Mar. 2010.