On disjoint crossing families in geometric graphs

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Abstract

A *geometric graph* is a graph drawn in the plane with vertices represented by points and edges as straight-line segments. A geometric graph contains a (k,l)-*crossing family* if there is a pair of edge subsets E_1, E_2 such that $|E_1| = k$ and $|E_2| = l$, the edges in E_1 are pairwise crossing, the edges in E_2 are pairwise crossing, and every edge in E_1 is disjoint to every edge in E_2. We conjecture that for any fixed k, l, every n-vertex geometric graph with no (k,l)-crossing family has at most $c_{k,l}n$ edges, where $c_{k,l}$ is a constant that depends only on k and l. In this note, we show that every n-vertex geometric graph with no (k,k)-crossing family has at most $c_k n \log n$ edges, where c_k is a constant that depends only on k, by proving a more general result which relates extremal function of a geometric graph F with extremal function of two completely disjoint copies of F. We also settle the conjecture for geometric graphs with no $(2,1)$-crossing family. As a direct application, this implies that for any circle graph F on 3 vertices, every n-vertex geometric graph that does not contain a matching whose intersection graph is F has at most $O(n)$ edges.

1 Introduction

A *topological graph* is a graph drawn in the plane with points as vertices and edges as non-self-intersecting arcs connecting its vertices. The arcs are allowed to intersect, but they may not pass through vertices except for their endpoints. Furthermore, the edges are not allowed to have tangencies, i.e., if two edges share an interior point, then they must properly cross at that point. We only consider graphs without parallel edges or self-loops. A topological graph is *simple* if every pair of its edges intersect at most once. If the edges are drawn as straight-line segments, then the graph is *geometric*. Two edges of a topological graph *cross* if their interiors share a point, and are *disjoint* if they do not have a point in common (including their endpoints).

It follows from Euler's Polyhedral Formula that every simple topological graph on n vertices and no crossing edges has at most $3n-6$ edges. It is also known that every simple topological graph on n vertices with no pair of disjoint edges has at most $O(n)$ edges [10],[7]. Finding the maximum number of edges in a topological (and geometric) graph with a forbidden substructure has been a classic problem in extremal topological graph theory (see [1], [2], [15], [6], [20], [14], [19], [18], [21]). Many of these problems ask for the maximum number of edges in a topological (or geometric) graph whose edge set does not contain a matching that defines a particular intersection graph. Recall that the *intersection graph* of objects $\mathcal{C}$ in the plane is a graph with vertex set $\mathcal{C}$, and two vertices are adjacent if their corresponding objects intersect. Much research has been devoted to understanding the clique and independence number of intersection graphs due to their applications in VLSI design [8], map labeling [3], and elsewhere.

Recently, Ackerman et al. [4] defined a *natural* (k,l)-*grid* to be a set of k pairwise disjoint edges that all cross another set of l pairwise disjoint edges. They conjectured

Conjecture 1 *Given fixed constants $k, l \geq 1$ there exists another constant $c_{k,l}$, such that any geometric graph on n vertices with no natural (k,l)-grid has at most $c_{k,l}n$ edges.*

They were able to show,

Theorem 2 [4] *For fixed k, an n-vertex geometric graph with no natural (k,k)-grid has at most $O(n \log^2 n)$ edges.*

Theorem 3 [4] *An n-vertex geometric graph with no natural $(2,1)$-grid has at most $O(n)$ edges.*

Theorem 4 [4] *An n-vertex simple topological graph with no natural (k,k)-grid has at most $O(n \log^{4k-6} n)$ edges.*

As a *dual* version of the natural (k,l)-grid, we define a (k,l)-*crossing family* to be a pair of edge subsets E_1, E_2 such that

1. $|E_1| = k$ and $|E_2| = l$,
2. the edges in E_1 are pairwise crossing,
3. the edges in E_2 are pairwise crossing,
4. every edge in E_1 is disjoint to every edge in E_2.

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We conjecture:

Conjecture 5 *Given fixed constants $k, l \geq 1$ there exists another constant $c_{k,l}$, such that any geometric graph on n vertices with no (k,l)-crossing family has at most $c_{k,l}n$ edges.*

It is not even known if all n-vertex geometric graphs with no k pairwise crossing edges has $O(n)$ edges. The best known bound is due to Valtr [22], who showed that this is at most $O(n \log n)$ for every fixed k. We extend this result to (k,k)-crossing families by proving the following theorem.

Theorem 6 *An n-vertex geometric graph with no (k,k)-crossing family has at most $c_k n \log n$ edges, where c_k is a constant that depends only on k.*

Let F denote a geometric graph. We say that a geometric graph G contains F as a geometric subgraph if G contains a subgraph F' isomorphic to F such that two edges in F' cross if and only if the two corresponding edges cross in F.

We define $ex(F,n)$ to be the extremal function of F, i.e. the maximum number of edges a geometric graph on n vertices can have without containing F as a geometric subgraph. Similarly, we define $ex_L(F,n)$ to be the extremal function of F, if we restrict ourselves to the geometric graphs all of whose edges can be hit by one line.

Let F_2 denote a geometric graph, which consists of two completely disjoint copies of a geometric graph F. We prove Theorem 6 by a straightforward application of the following result.

Theorem 7 $ex(F_2,n) = O((ex_L(F,2n)+n)\log n + ex(F,n))$

Furthermore, we settle Conjecture 5 in the first nontrivial case.

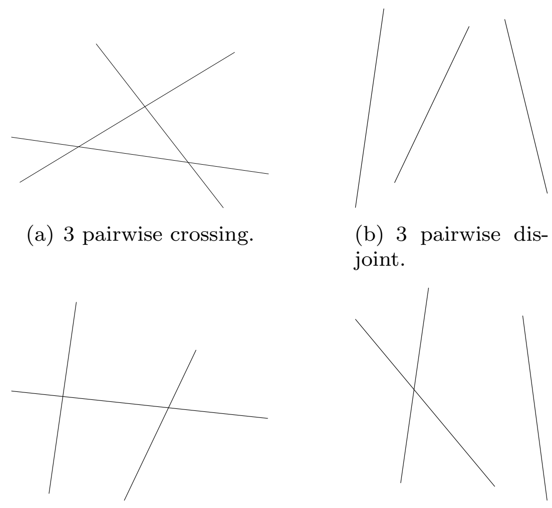
Theorem 8 *An n-vertex geometric graph with no $(2,1)$-crossing family has at most $O(n)$ edges.*

Note that Conjecture 5 is not true for topological graphs since Pach and Tóth [16] showed that the complete graph can be drawn such that every pair of edges intersect once or twice.

Recall that F is a *circle graph* if F can be represented as the intersection graph of chords on a circle. By combining Theorem 8 with results from [2], [4], and [20], we have the following.

Corollary 9 *For any circle graph F on 3 vertices, every n-vertex geometric graph that does not contain a matching whose intersection graph is F contains at most $O(n)$ edges.*

See Figure 1. We also conjecture the following.



(a) 3 pairwise crossing. (b) 3 pairwise disjoint. (c) (2,1)-grid. (d) (2,1)-crossing family.

Figure 1: Triples of segments corresponding to all circle graphs on three vertices.

Conjecture 10 *For any circle graph F on k vertices, there exists a constant c_k such that every n-vertex geometric graph that does not contain a matching whose intersection graph is F, contains at most $c_k n$ edges.*

As pointed out by Klazar and Marcus [9], it is not hard to modify the proof of the Marcus-Tardos Theorem [19] to show that Conjecture 10 is true when the vertices are in convex position.

For simple topological graphs, we have the following

Theorem 11 *An n-vertex simple topological graph with no $(k,1)$-crossing family has at most $n(\log n)^{O(\log k)}$ edges.*

2 Relating extremal functions

First, we prove a variant of Theorem 7 when all of the edges in our geometric graph can be hit by a line. As in the introduction let F_2 denote a geometric graph, which consists of two completely disjoint copies of a geometric graph F. We will now show that the extremal function $ex_L(F_2,n)$ is not far from $ex_L(F,n)$.

Theorem 12 *$ex_L(F_2,n) \leq O((n+ex_L(F,2n))\log n)$*

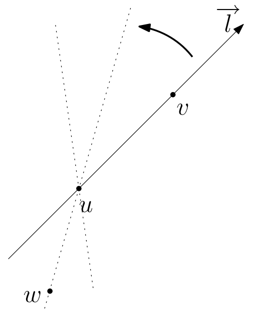
Proof. Let G denote a geometric graph on n vertices that does not contain F_2 as a geometric subgraph, and all the edges of G can be hit by a line. By a standard perturbation argument we can assume that the vertices of G are in general position. As in [5], a *halving edge* uv is a pair of the vertices in G such that the number of vertices on each side of the line through u and v is the same.

Lemma 13 *There exists a directed line $\vec{l}$ such that the number of edges in G that lies completely to the left or right of $\vec{l}$ is at most $2ex_L(F, n/2) + 5n$.*

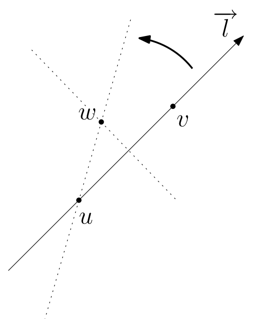
Proof. If n is odd we can discard one vertex of G, thereby loosing at most n edges. Therefore we can assume n is even, and it suffices to show that there exists a directed line $\vec{l}$ such that the number of edges in G that lies completely to the left or right of $\vec{l}$ is at most $2ex_L(F, n/2) + 4n$.

Let uv be a halving edge, and let $\vec{l}$ denote the directed line containing vertices u and v with direction from u to v. Let $e(\vec{l}, L)$ and $e(\vec{l}, R)$ denote the number of edges on the left and right side of $\vec{l}$ respectively. Without loss of generality, we can assume that $e(\vec{l}, L) \leq e(\vec{l}, R)$. We will rotate $\vec{l}$ such that it remains a halving line at the end of each step, until it reaches a position where the number of edges on both sides of $\vec{l}$ is roughly the same.

We start by rotating $\vec{l}$ counterclockwise around u until it meets the next vertex w of G. If initially w lies to the right of $\vec{l}$, then in the next step we will rotate $\vec{l}$ around u (again). See Figure 2(a). Otherwise if w was on the left side of $\vec{l}$, then in the next step we will rotate $\vec{l}$ around vertex w. See Figure 2(b). Clearly after each step in the rotation, there are exactly $n/2$ vertices on each side of $\vec{l}$.



(a) w lies to the right of $\vec{l}$.



(b) w lies to the left of $\vec{l}$.

Figure 2: Halving the vertices of G

After several rotations, $\vec{l}$ will eventually contain points u and v again, with direction from v to u. At this point we have $e(\vec{l}, L) \geq e(\vec{l}, R)$. Since the number of edges on the right side (and left side) changes by at most n after each step in the rotation, at some point in the rotation we must have

$$|e(\vec{l}, L) - e(\vec{l}, R)| \leq 2n.$$

Since G does not contain a F_2 as a geometric subgraph, this implies that

$$e(\vec{l}, L) \leq ex_L(F, n/2) + 2n$$

and

$$e(\vec{l}, R) \leq ex_L(F, n/2) + 2n.$$

Therefore for any n, there exists a directed line $\vec{l}$ such that the number of edges in G that lies completely to the left or right of $\vec{l}$ is at most $2ex_L(F, n/2) + 5n$. □

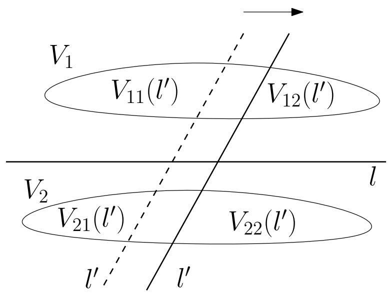


Figure 3: The final partition of the vertex set of G

By Lemma 13 we obtain a line l, which partition the vertices of G into two equal (or almost equal if n is odd) sets V_1 and V_2. Let E' denote the set of edges between V_1 and V_2. By the Ham-Sandwich Cut Theorem [11], there exists a line l' that simultaneously bisects V_1 and V_2. Let $V_{11}(l')$ and $V_{12}(l')$ denote the resulting parts of V_1, and let $V_{21}(l')$ and $V_{22}(l')$ denote the resulting parts of V_2.

Observe that we can translate l' along l into a position where the number of edges in E' that lie completely to the left and completely to the right of l' is roughly the same. In particular, we can translate l' along l such that the number of edges in E' that lies completely to its left or right side is at most $ex_L(F, n) + ex_L(F, n/2+1) + n$ (see Figure 2). Indeed, assume that the number of edges in E' between, say, $V_{12}(l')$ and $V_{22}(l')$ is more than $ex_L(F, n/2 + 1)$. As we translate l' to the right, the number of edges that lie completely to the right of l' changes by at most n as l' crosses a single vertex in G. Therefore we can translate l' into the leftmost position where the number of edges in E' between $V_{12}(l')$ and $V_{22}(l')$ drops below $ex_L(F, n/2 + 1) + n + 1$. Moreover, at this position the number of edges in E' between $V_{11}(l')$ and $V_{21}(l')$ still cannot be more than $ex_L(F, n)$ since G does not contain F_2 as a geometric subgraph.

Thus, all but at most $3ex_L(F, n/2 + 1) + ex_L(F, n) + 6n$ edges of G are the edges between $V_{11}(l')$ and $V_{22}(l')$, and between $V_{12}(l')$ and $V_{21}(l')$. Notice that there exists k, $-1/4 \leq k \leq 1/4$, such that $|V_{11}(l')| + |V_{22}(l')| = n(1/2 + k)$, and $|V_{12}(l')| + |V_{21}(l')| = n(1/2 - k)$. Finally, we are in the position to state the recurrence, whose closed form gives the statement of the theorem:

$$ex_L(F_2, n) \leq ex_L(F_2, n(1/2 + k)) + ex_L(F_2, n(1/2 - k)) + 3ex_L(F, n/2 + 1) + ex_L(F, n) + 6n.$$

□

Finally, we show how Theorem 12 implies Theorem 7.

Proof. [Proof of Theorem 7.] Let $G = (V, E)$ denote the geometric graph not containing F_2 as a subgraph. Similarly, as in the proof of Lemma 13 we can find a halving line l that hits all but $2ex(F, n/2)+5n$ edges of G. Now, the claim follows by using Theorem 12. □

Theorem 6 follows easily by using Theorem 7 with a result from [22], which states that every n-vertex geometric graph whose edges can be all hit by a line and does not contain k pairwise crossing edges has at most $O(n)$ edges and at most $O(n \log n)$ edges if we do not require a single line to hit all the edges.

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