Data Imprecision under λ -Geometry: Finding the Largest Axis-Aligned Bounding Box

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Abstract

In this paper we introduce a new model for handling imprecision in the input data of a geometric problem. The proposed model, which is called λ-*geometry*, is a generalization of region based models to handle dynamic imprecision. Further, we study the problem of finding the largest area axis-aligned bounding box of a set of n imprecise points under the model. We propose an $O\big(n(\log n + \log k + m)\big)$ time algorithm to solve this problem, where k is the maximum complexity of the regions representing imprecise points, and m is the maximum number of corner defining functions when the largest axis-aligned bounding box is defined by points on its corners.

1 Introduction

Classical computational geometry focuses mainly on problems whose input data has precise location. Although these problems have their own interest, they are not very close to what we face in the real world. In the real world, input data is collected by using devices with limited precision. Therefore, imprecision in the location of input data is inevitable. In order to take imprecision into account, several models have been proposed. Region based models are the most common types [5]. Once a model can represent imprecise points, it can be generalized to represent imprecise lines and polygons as well. So, the initial matter is how to model an imprecise point. In a region based model, location of each imprecise point is assumed to be anywhere in a predefined region. The model of ε-geometry [3] is of the earliest models of this type. In this model each point can have an imprecision up to ε in any direction, and as a result regions are assumed to be disks. Other regions such as segments, rectangles, and convex polygons have also been proposed to represent imprecision. Further, several computational geometry problems have been studied under region based models, such as finding the largest/smallest area axis-aligned bounding box [6], and convex hull [5]. A thorough study is presented in [5].

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Besides solving computational geometry problems under a specific model of imprecision, developing more general and practical models for imprecision is of great importance. Proposing a practical model that deals with dependencies among imprecise input data is of the recent work done in this field [4].

As reviewed above, region based models have a static view on the level of imprecision. That is once the regions that represent imprecision are known; they stay unchanged throughout the algorithm. However, level of imprecision can vary continuously according to circumstances such as changes in the precision of measurement. So, in this paper we introduce a new model that can handle dynamic level of imprecision. Let λ represent the level of imprecision, where $\lambda \in [0, 1]$. Our proposed model which is called λ-*geometry* is a generalization of region based models that allows regions to be growing or shrinking according to increase or decrease in λ in a monotone continuous manner. This model also provides an inherent dependency in the level of imprecision of the input data.

The paper is organized as follows. In Section 2, we define λ-geometry more precisely. Later, in section 3 we show how the problem of finding the maximum area axis-aligned bounding box of a set of imprecise points can be solved under the model of λ-geometry.

2 The model of λ-Geometry

We begin introducing the model of λ-geometry by defining an imprecise point in it. An imprecise point p in this model is defined as $p = \big(\overline{p}, M, \lambda\big)$, where $\overline{p}$ is the *exact value* of the imprecise point p, $M = [v_1, v_2, ..., v_k]_{2\times k}$ is the *imprecision matrix* in which each vector v_i defines the maximum imprecision in its direction, and the parameter λ shows the *imprecision level* for each v_i. So, for any $\lambda \in [0, 1]$, a region is considered for modeling an imprecise point. This region, which includes all possible instances of a point p, is the convex hull of points defined by the sum of the vectors λv_i and $\overline{p}$. Fig.1(a) illustrates a sample imprecise point p for two different λs, $\lambda_1 = 2/3$ and $\lambda_2 = 1$, where $\overline{p} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $M = \begin{bmatrix} 2 & -2 & -3 \\ 0 & 3 & -2 \end{bmatrix}$.

It is obvious that for $\lambda = 1$ an imprecise point p is the convex hull of the points inducing by the vectors of imprecision matrix M, while for $\lambda = 0$ this imprecise

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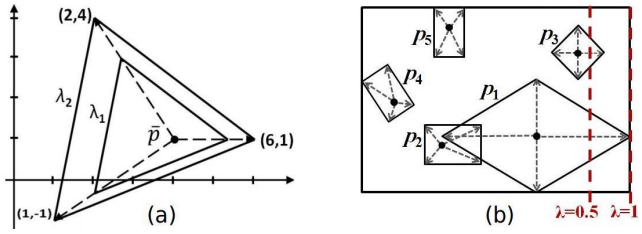

Figure 1: (a) An imprecise point p for $\lambda_1 = 2/3$ and $\lambda_2 = 1$. (b) The AABB constructed with extreme imprecise points in each of the four axis-aligned directions. For $\lambda = 1$, p_1 is the rightmost, while for $\lambda = 0.5$, p_3 is the rightmost imprecise point.

point is just the exact point $\overline{p}$ [1].

Now, we focus on algebraic definition of an instance of an imprecise point, $p(\lambda, \gamma)$:

$$p(\lambda,\gamma) \stackrel{\text{def}}{=} \overline{p} + \lambda.M\gamma \ ; \quad \gamma = [\gamma_1, \gamma_2, \cdots, \gamma_k]^T \ ,$$
$$, 0 \le \gamma_i \le 1, \quad \sum_{i=1}^{k} \gamma_i \le 1.$$

In this definition γ is an arbitrary vector that defines an instance $p(\lambda, \gamma)$ to be anywhere in the convex hull mentioned earlier. So, an imprecise point, $p(\lambda)$, is a region defined by

$$p(\lambda) \stackrel{\text{def}}{=} \big\{p(\lambda,\gamma) \mid \forall\gamma \ ; \ 0 \le \gamma_i \le 1, \sum_{i=1}^{k} \gamma_i \le 1\big\}.$$

In many models of imprecise data, a line defined by two imprecise points is the union of lines that passes through the both corresponding regions [5]. Similarly, a line in the model of λ-geometry is defined as follows:

$$L_{p,q}(\lambda) \stackrel{\text{def}}{=} \big\{L(p', q') \mid p', q' \text{ are instances of } p(\lambda), q(\lambda)\big\}.$$

This definition can be generalized for defining imprecise polygons.

3 Largest Axis-Aligned Bounding Box

Given a set P of n points in the plane, the axis-aligned bounding box (AABB) of P is the smallest axis-aligned rectangle that contains P. The key step to find the AABB of P is to find the extreme point of P in each of the four axis-aligned directions. In an imprecise context when imprecise points are modeled by regions, choosing an instance from each imprecise point results in an AABB. So in this case, the problem is where to choose an instance from each imprecise point to make the area of AABB of these instances maximized for example, (*the largest AABB problem*). From now on we use AABB to refer to the largest AABB.

In the model of λ-geometry, since regions are growing (or shrinking) with the rate of λ, it is possible for each region to be extreme in an axis-aligned direction for a specific $\lambda \in [0, 1]$, see Fig.1(b). So, to find the AABB in the model of λ-geometry, the first step is to break the interval of $[0, 1]$ in some subintervals for which the four extreme regions stay unchanged. Thus, the output will be as follows: $(\lambda_{i-1}, \lambda_i, f_{Ri}, f_{Li}, f_{Ti}, f_{Bi})$, such that $1 \le i \le n'$, $\lambda_i \in [0, 1]$, $\lambda_i < \lambda_{i+1}$, and f_{Ri}, f_{Li}, f_{Ti} and f_{Bi} are functions defining respectively the right, left, top and bottom side of the AABB in the interval of $[\lambda_{i-1}, \lambda_i]$. Next we explain how to find these values.

Consider the imprecise point $p_i(\lambda) = (\overline{p_i}, M_i, \lambda)$, where $\overline{p_i} = (\overline{x_i}, \overline{y_i})$. The rightmost point of $p_i(\lambda)$ is defined by the vector of $v_{iR} = (a_{iR}, b_{iR})$ which is the rightmost vector of M_i. Thus, the rightmost point of $p_i(\lambda)$ can be defined by a linear function $R_i(\lambda) = \overline{x_i} + \lambda a_{iR}$. We call $R_i(\lambda)$ the *rightmost defining function* of $p_i(\lambda)$. Since $p_i(\lambda)$ is a convex polygon, a_{iR} can be found in $O(\log k_i)$ time, where $k_i = |M_i|$ and $|M_i|$ denotes the number of columns in the matrix M_i.

Given a set of imprecise points, let $R = \{R_1, R_2, \cdots, R_n\}$ be the set of all rightmost defining functions. So, the right side of the AABB can be determined by the upper envelope of R. Functions of R are totally defined in $[0, 1]$, and each pair of these functions intersects in at most one point. Thus, the upper envelope of R has at most n breakpoints, and it can be computed in $O(n \log n)$ time [7]. The upper envelopes of the left, top, and bottommost defining functions can be computed similarly. So after computing these four upper envelopes, the union of their breakpoints constructs the critical λs of this step. Let $k = \max_{1 \le i \le n} k_i$. Then the algorithm proposed to find these critical λs runs in $O(n \log k + n \log n)$ time. When k is polynomial in n, we prove this algorithm is optimal.

Lemma 1 *Given a set of n imprecise points in the model of λ-geometry, finding all the rightmost defining functions for the largest AABB problem requires $\Omega(n \log n)$ computations in the worst case.*

Proof. The proof is by a reduction from the sorting problem. Let $S = \{s_1, s_2, \cdots, s_n\}$ be a set of n positive real numbers, and let d be a number greater than the biggest difference of numbers in S. We convert each number s_i to an imprecise point $p_i = (\overline{p_i}, M_i, \lambda)$ with a symmetric rhomboid as its region. We define the rightmost vector of p_i to be $R_i = m_i\lambda - m_i s_i$, where $m_i = s_i/(d - s_i)$. Based on the formula of the rightmost defining function, R_i can be seen as the vector of $V_{iR} = \begin{bmatrix} m_i \\ 0 \end{bmatrix}$ which starts at the point $\overline{p_i} = (-m_i s_i, 0)$. Similarly, we define the left, top and bottom vectors and finally we obtain the imprecision matrix $M = \begin{bmatrix} m_i & 0 & -m_i & 0 \\ 0 & m_i & 0 & -m_i \end{bmatrix}$. We claim that the order in which the rightmost defining functions of the constructed imprecise points appear on the upper envelope is the same as the order of sorted numbers.

[1] Note that since we define p by the concept of convex hull, it is possible for a vector to be ineffective. But without loss of generality, we assume that all vectors are effective.

To prove this it is sufficient to show that for any three positive numbers s_i, s_j, s_k, where $s_i < s_j < s_k$, $R_i(\lambda)$ appears before $R_j(\lambda)$ on the upper envelope, and if x_{ij} (resp. x_{ik}) is the intersection point of $R_i(\lambda)$ and $R_j(\lambda)$ $\big($resp. $R_i(\lambda)$ and $R_k(\lambda)\big)$, then $x_{ij} < x_{ik}$. This is a straightforward result. □

In finding critical λs and the corresponding right, left, top and bottommost defining functions there is still one issue left to handle, and that is we can not choose more than one instance from each imprecise point. For a fixed value of λ, Fig.1(b) illustrates a situation where the right and the bottom side of the AABB is defined by two different instances of an imprecise point $p_1(\lambda)$. To handle this situation either we should take just the rightmost or just the bottommost instance of $p_1(\lambda)$, or we should take an instance of $p_1(\lambda)$ that plays these roles together that is the instance taken should be the lower-right corner of the AABB. The final choice is the one that leads to a larger area of AABB. Next we explain how to handle this issue in the model of λ-geometry.

For a fixed value of λ, the AABB of a set of n imprecise points modeled by regions can be determined by testing only the four farthest imprecise points in each of the four axis-aligned directions [6]. Further, let $F = \{f_1, f_2, \cdots, f_n\}$ be an arrangement of n lines in the plane and let c be a constant. It is known that the c^{th}-upper level of F, denoted by E_F^c, has $O(n)$ breakpoints and it can be computed in $O(n \log n)$ time [2]. So, in the model of λ-geometry to find the AABB determined by four instances of different imprecise points we do the following. First we compute the set of rightmost, the set of leftmost, the set of topmost and the set of bottommost defining functions. Let R, L, T, and B denote these sets respectively. Then, for each of these sets we find c^{th}-upper level for $c = 1, 2, 3, 4$. Let E_R^c, E_L^c, E_T^c and E_B^c denote these upper levels for the sets R, L, T, and B respectively. Now, we define the set of critical λs as:

$\Lambda = \big\{\lambda | \lambda$ is a break point for E_R^c or E_L^c or E_T^c or E_B^c for $c = 1, 2, 3, 4\big\}$

Any two consecutive members of the ordered set Λ construct an interval in which none of the four upper levels changes. So, in each interval constructed if the first upper level of each set is different from the first upper levels of the other sets, we report this interval and its corresponding upper levels (in this case we are sure that four different imprecise points have constructed the AABB). Otherwise, if the first upper levels of some sets are equal, we take the first upper level of a set and test the second, the third, and finally in the worst case the forth upper levels of the remaining sets, and we select the one that leads to a larger area of AABB. So, by the approach explained we can find the AABB that is determined by four instances of different imprecise points. Further, when one or two instances play the role of corners of the AABB, three or two instances of different imprecise points can also determine the AABB. So, we should compute the AABB of these situations as well, and at the end report the largest AABB we have found over all situations. The case of three instances occurs when one corner of the AABB is constructed by two equal upper levels (for example when $E_L^1 \neq E_T^1$ and $E_R^1 = E_B^1$). Further, the case of two instances occurs when two opposite corners of the AABB are constructed by equal upper levels (for example when $E_L^1 = E_T^1$ and $E_R^1 = E_B^1$). So, the final step of the algorithm is how to find the AABB of these two cases. Next we show how to handle this.

Let $p(\lambda)$ be an imprecise point whose instance is a corner of the AABB. Obviously this instance should be taken from the boundary of $p(\lambda)$. Thus, the boundary of $p(\lambda)$, and more precisely, a specific part of this boundary called a chain is important to us. Assume that vertices of the boundaries of the imprecise points are in a clockwise order. We call a chain *convex* (resp. *concave*) if it is xy-monotone and it lies above (resp. below) any line that passes through its segments.

Lemma 2 *Let $C_p = \{p_1, p_2, \cdots, p_k\}$ be a concave chain in the second quadrant of the coordinate system, while $C_q = \{q_1, q_2, \cdots, q_k\}$ is a convex chain in the forth quadrant. Then the AABB whose corners lie on C_p and C_q can be computed in $O(\log k)$ time.*

Proof. It is known that the largest inscribed isothetic rectangle inside a k-sided polygon can be computed by an $O(\log k)$ time algorithm [1]. We can treat the two chains' problem as a special case of this algorithm. Thus, the AABB whose corners lie on C_p and C_q can be computed in $O(\log k)$ time as well. □

Lemma 3 *Let $Cor(\lambda) = (\lambda a + a', \lambda b + b')$ represent the upper-left corner of the AABB and let $q = (\overline{q}, M_q, \lambda)$ be the rightmost and bottommost imprecise point. Then, the AABB can be computed in $O(\log k + m)$ time, where $k = |M_q|$ and m is the number of functions defining the lower-right corner of the AABB, while λ decreases from 1 to 0.*

The proof is omitted due to space limitations. Since lemma 3 is a special case of lemma 4, we refer readers to see the proof of lemma 4.

So far we have shown how to find the AABB constructed by four instances, and the AABB constructed by three instances of imprecise points. The following lemma shows how to find the AABB constructed by two instances of imprecise points.

Lemma 4 *Let $p = (\overline{p}, M_p, \lambda)$ be the leftmost and topmost, and let $q = (\overline{q}, M_q, \lambda)$ be the rightmost and bottommost imprecise points. Then the AABB can be computed in $O(\log k + m)$ time, where $k =$*

$\max\big(|M_p|, |M_q|\big)$ *and* m *is the number of functions defining corners, while* λ *decreases from* 1 *to* 0.

Proof. It is known that in the case of two segments, at least one corner of the AABB lies on endpoints of these segments [1]. So, for any fixed value of λ, at least one vector of M_p and one edge of M_q (or vice versa) define the AABB. This is still true while points are growing or shrinking in the model of λ-geometry. This fact helps us to find the optimal path for each of two corners of the AABB. To do so, it is sufficient to find critical λs for which the vector or the segment that defines the AABB changes. Let $l'\big(a(\lambda), s(\lambda)\big)$ denote the optimal path for the point $a(\lambda)$ and the segment $s(\lambda)$. The parametric equation of $l'\big(a(\lambda), s(\lambda)\big)$ can be obtained by computing the area function and its first derivative [2]. First for $\lambda = 1$ we compute two corners of the AABB by using lemma 2. These corners are taken from p and q, and are denoted by c_p and c_q respectively. Set $\lambda_c = 1$. According to the position of these corners two cases arise:

Case1: The corner c_p lies on a vertex of $p(\lambda)$, while the corner c_q lies on a segment of $q(\lambda)$(or vice versa). We denote this vertex of $p(\lambda)$ by p_i, and this segment of $q(\lambda)$ by $s_j^q = \overline{q_j q_{j+1}}$ (see Fig.2 for $0.7 \le \lambda \le 1$ and $0.2 \le \lambda \le 0.3$). In this case, we take vectors v_j and v_{j+1} of M_q, and find their intersection with line $l'(c_p, s_j^q)$. Note that only one of these vectors intersects $l'(c_p, s_j^q)$. Then we compute the value of λ corresponding to this intersection, and denote it by λ_q. Further, we take the vector v_i of M_p and find its intersection with $l'(v_j, s_i^p)$ and $l'(v_j, s_{i+1}^p)$ (just one of these lines intersects v_i). Let λ_p be the value of λ corresponding to this intersection. Set $\lambda_n = \max(\lambda_p, \lambda_q)$ and report two functions v_i and $l'(c_p, s_j^q)$ as two corners' paths in $[\lambda_n, \lambda_c]$. If $\lambda_n \le 0$ the algorithm is finished. Otherwise, set $\lambda_c = \lambda_n$ and go to case 2.

Case2: Both corners c_p and c_q lie on vertices of $p(\lambda)$ and $q(\lambda)$. Let p_i denote the corresponding vertex of $p(\lambda)$, and let q_j denote the corresponding vertex of $q(\lambda)$ (see Fig. 2 for $0.3 \le \lambda \le 0.7$ or $0 \le \lambda \le 0.2$). In this case, we take vector v_j of M_q, and find its intersection with $l'(v_i, s_j^q)$ and $l'(v_i, s_{j+1}^q)$. Let λ_q denote the value of λ corresponding to this intersection. Further, we take vector v_i of M_p, and compute its intersection with $l'(v_j, s_i^p)$ and $l'(v_j, s_{i+1}^p)$. Let λ_p denote the value of corresponding λ. Set $\lambda_n = \max(\lambda_p, \lambda_q)$ and report two functions v_i and v_i as two corners' paths in $[\lambda_n, \lambda_c]$. If $\lambda_n \le 0$ the algorithm is finished. Otherwise, set $\lambda_c = \lambda_n$ and go to case 1.

In each iteration, the algorithm reports an interval and two functions defining the optimal paths of the upper left and lower right corners within this interval. Let m denote the total number of corner defining functions in $[0, 1]$. Then the algorithm iterates m times.

[2]Note that while λ decreses from 1 to 0, $a(\lambda)$ is a segment while $s(\lambda)$ is a triangle.

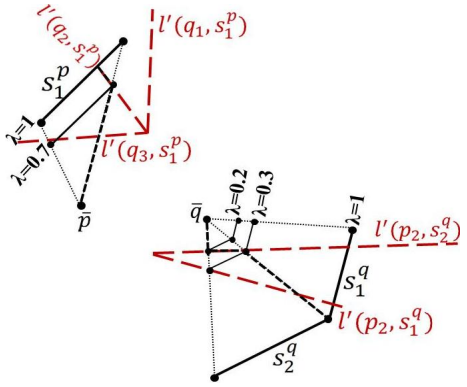

Figure 2: Concepts used in finding corner defining functions. Short dashes show the optimal paths for the corners.

Finding the first optimal corners (for $\lambda = 1$) takes $O(\log k)$ time by using lemma 2, but since we have these corners in the following iterations, each of the following iterations takes constant time. Therefore, the total time complexity is $O(\log k + m)$. Note that $m = O(k)$. □

Thus, in the model of λ-geometry we have obtained the following results:

Theorem 5 *Given a set of* n *imprecise points under the model of* λ*-geometry, the largest AABB problem can be solved in* $O\big(n(\log n + \log k + m)\big)$ *time, where* k *is the maximum complexity of regions representing imprecision, and* m *is the maximum number of corner defining functions.*

Note that when $k = O(\log n)$, the largest AABB problem under the model of λ-geometry can be solved in the optimal time $\Theta(n \log n)$.

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