Connecting a Set of Circles with Minimum Sum of Radii

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Abstract

We consider the problem of assigning radii to given points in the plane, such that the resulting set of circles is connected, and the sum of radii is minimized. We show that the problem is polynomially solvable if a connectivity tree is given, but NP-hard if there are upper bounds on the radii; the case of unbounded radii is an open problem. We also give approximation guarantees for a variety of heuristics, describe upper and lower bounds (which are matching in some of the cases), and conclude with experimental results.

1 Introduction

We consider a natural geometric connectivity problem, arising from the context of assigning ranges to a set of center points. More formally, given a set of points $P = \{p_1, \ldots, p_n\}$ in the plane and their respective radii r_i, in the connectivity graph $G = \{V, E\}$, V corresponds to P, and edges $e_{ij} \in E$ to intersecting circles, i.e., $r_i + r_j \geq \text{dist}(p_i, p_j)$ for the Euclidean distance $\text{dist}(p_i, p_j)$ between p_i and p_j. (A natural generalization arises from considering distances in a given weighted graph, instead of geometric distances.) The Connected Range Assignment Problem (CRA) requires assignment of radii r to P, such that the objective function $Q = \sum_{i=1}^{n} r_i^{\alpha}, \alpha = 1$ is minimized, subject to the constraint that G is connected.

Problems of this type have been considered before and have natural motivations from fields including networks, robotics, and data analysis. Common to most is an objective function that considers the sum of the radii of circles to some exponent α.

Alt *et al.* [1] consider the closely related problem of selecting circle centers and radii such that a given set of points in the plane are covered by the circles. Like our work, they focus on minimizing an objective function based on $\sum_i r_i^{\alpha}$ and produce results specific to various values of α. The minimum sum of radii circle coverage problem (with $\alpha = 1$) is also considered by Lev-Tov and Peleg [6] in the context of radio networks. Related work has also been done in the area of data clustering. Gibson *et al.* [5], consider partitioning data into k clusters to minimize the sum of the cluster radii, and authors consider the problem for specific numbers of dimensions. Since we are given the circle centers, the problem can be also considered a *range assignment problem* [3]; see [4] for hardness results of different (typically directed) *communication graphs*.

In this paper we present a variety of algorithmic aspects of the problem. In Section 2 we show that for a given connectivity tree, an optimal solution can be computed efficiently. Section 3 sketches a proof of NP-hardness for the problem when there is an upper bound on the radii. Section 4 provides a number of approximation results for the case of unbounded radii, complemented by experiments in Section 5.

2 CRA for a Given Connectivity Tree

For a given connectivity tree, our problem is polynomially solvable, based on the following observation.

Lemma 1 *Given a connectivity tree T with at least three nodes. There exists an optimal range assignment for T with $r_i = 0$ for all leaves p_i of T.*

Proof. Assume an optimal range assignment for T has a leaf $p_i \in P$ with radius $r_i > 0$. The circle C_i around p_i intersects circle C_j around p_i's parent p_j with radius r_j. Extending C_j to $r_j := \text{dist}(p_i, p_j)$ while setting $r_i := 0$ does not increase $\sum_{p_i \in P} r_i$. □

Direct consequences of Lemma 1 are the following.

Corollary 2 *There is an optimal range assignment satisfying Lemma 1 and further $r_j > 0$ for all $p_j \in P$ of height 1 in T (i.e., each p_j is parent of leaves only).*

Corollary 3 *Consider an optimal range assignment for T satisfying Lemma 1. Further let $p_j \in P$ be of height 1 in T. Then $r_j \geq \max_{p_i \text{ is child of } p_j} \{dist(p_i, p_j)\}$.*

These observations allow us to solve the problem via dynamic programming; details are omitted.

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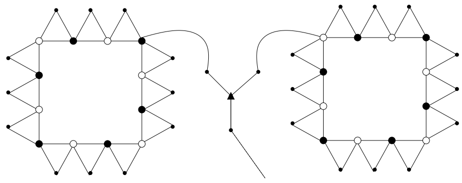

Figure 1: Two variable gadgets connected to the same clause gadget. "True" and "False" vertices marked in bold white or black; auxiliary vertices are indicated by small dots; the clause vertex is indicated by a triangle. Connectivity edges are not shown.

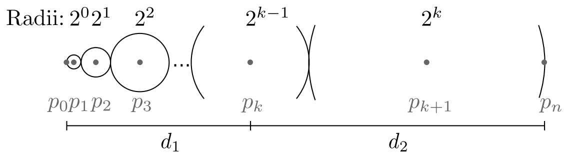

Figure 2: A class of CRA instances that need $k+1$ circles in an optimal solution.

Theorem 4 *For a given connectivity tree, CRA can be solved in polynomial time.*

3 Range Assignment for Bounded Radii

Without a connectivity tree, and assuming an upper bound of ρ on the radii, the problem becomes NP-hard; in this short abstract, we focus on the graph version of the problem.

Theorem 5 *With radii bounded by some constant ρ, the problem CRA is NP-hard in weighted graphs.*

See Figure 1 for the basic construction. The proof uses a reduction from 3SAT. Variable are represented by closed "loops" at distance ρ that have two feasible connected solutions: auxiliary points ensure that either the odd or the even points in a loop get radius ρ. Additional "connectivity" edges ensure that all variable gadgets are connected. Each clause is represented by a star-shaped set of four points that is covered by one circle of radius ρ from the center point. This circle is connected to the rest of the circles, if and only if one of the variable loop circles intersects it, which is the case if and only if there is a satisfying variable.

4 Solutions with a Bounded Number of Circles

In this section we show that using only a small number of circles already yields good approximations; we start by a class of lower bounds.

Theorem 6 *Even for a set of collinear points, a best k-circle solution may be off by a factor of $(1+\frac{1}{2^{k+1}})$.*

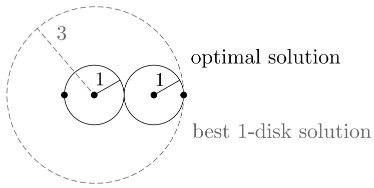

Figure 3: A lower bound of $\frac{3}{2}$ for 1-circle solutions.

Proof. Consider the example in Fig. 2. The provided solution is optimal, as $\sum r_i = \frac{\text{dist}(p_0,p_n)}{2}$. Further, for any integer $k \geq 2$ we have $d_1 = 2\sum_{i=0}^{k-2} 2^i + 2^{k-1} < 2 \cdot 2^k + 2^{k-1} = d_2$. So the radius r_{k+1} cannot be changed in an optimal solution. Inductively, we conclude that exactly $k+1$ circles are needed. Because we only consider integer distances, a best k-circle solution has cost $R_k \geq R+1$, i.e., $\frac{R_k}{R} \geq 1 + \frac{1}{2^{k+1}}$. □

In the following we give some good approximation guarantees for CRA using one or two circles.

Lemma 7 *Let $\mathcal{P}$ a longest path in an optimal connectivity graph, and let e_m be an edge in $\mathcal{P}$ containing the midpoint of $\mathcal{P}$. Then $\sum r_i \geq \max\{\frac{1}{2}|\mathcal{P}|, |e_m|\}$.*

Theorem 8 *A best 1-circle solution for CRA is a $\frac{3}{2}$-approximation.*

Proof. Consider a longest path $\mathcal{P} = (p_0, \ldots, p_k)$ of length $|\mathcal{P}|$ in an optimal connectivity graph. Let $R = \sum r_i$ be the cost of the optimal solution, and $e_m = p_i p_{i+1}$ as in Lemma 7. Let $\bar{d}_i := \text{dist}(p_i, p_k)$ and $\bar{d}_{i+1} := \text{dist}(p_0, p_{i+1})$. Then $\min\{\bar{d}_i, \bar{d}_{i+1}\} \leq \frac{\bar{d}_i + \bar{d}_{i+1}}{2} = \frac{\text{dist}(p_0,p_i)+2|e_m|+\text{dist}(p_{i+1},k)}{2} = \frac{|\mathcal{P}|}{2} + \frac{|e_m|}{2} \leq R + \frac{R}{2} = \frac{3}{2}R$. So one circle with radius $\frac{3}{2}R$ around the point in P that is nearest to the middle of path $\mathcal{P}$ covers P, as otherwise there would be a longer path. □

Fig. 3 shows that this bound is tight. Using two circles yields an even better approximation factor.

Theorem 9 *A best 2-circle solution for CRA is a $\frac{4}{3}$-approximation.*

Proof. Let $\mathcal{P} = (p_0, \ldots, p_k)$ be a longest path in an optimal connectivity graph. Then $\sum r_i \geq \frac{1}{2}|\mathcal{P}|$. We distinguish two cases; see Fig. 4.

Case 1. There is a point x on $\mathcal{P}$ at a distance of at least $\frac{1}{3}|\mathcal{P}|$ from both endpoints. Then there is a 1-circle solution that is a $\frac{4}{3}$-approximation, and no 2-circle solution of such quality is needed.

Case 2. There is no such point x. Let $e_m = p_i p_{i+1}$ be defined as in Lemma 7. Further, let $d_i := \text{dist}(p_0, p_i)$ and $d_{i+1} := \text{dist}(p_{i+1}, p_k)$. Then $|e_m| = |\mathcal{P}| - d_i - d_{i+1}$ and $d_i, d_{i+1} < \frac{1}{3}|\mathcal{P}|$.

Case 2a. If $|e_m| < \frac{1}{2}|\mathcal{P}|$ then $d_i + d_{i+1} = |\mathcal{P}| - |e_m| > \frac{1}{2}|\mathcal{P}| > |e_m|$. Set $r_i := d_i$ and $r_{i+1} = d_{i+1}$,

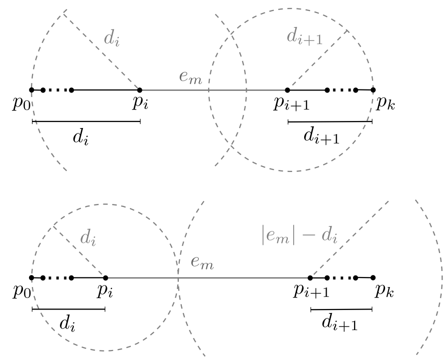

Figure 4: A best 2-circle solution is a $\frac{4}{3}$-approximate solution: Case 2a (Top); Case 2b (Bottom).

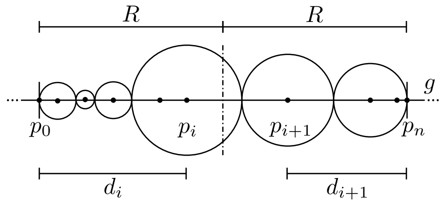

Figure 5: A non-overlapping optimal solution.

then the path is covered. Since $d_i, d_{i+1} < \frac{1}{3}|\mathcal{P}|$ we have $r_i + r_{i+1} = d_i + d_{i+1} < \frac{2}{3}|\mathcal{P}| \leq \frac{4}{3}\sum r_i$ and the claim holds.

Case 2b. Otherwise, if $|e_m| \geq \frac{1}{2}|\mathcal{P}|$ then $d_i + d_{i+1} \leq \frac{1}{2}|\mathcal{P}| \leq |e_m|$. Assume $d_i \geq d_{i+1}$. Choose $r_i := d_i$ and $r_{i+1} := |e_m| - d_i$. Then $\mathcal{P}$ is covered and $r_i + r_{i+1} = d_i + (|e_m| - d_i) = |e_m|$, which is the lower bound and thus the range assignment is optimal. □

If all points of P lie on a straight line, the approximation ratio for two circles can further be improved.

Lemma 10 *Let P be a subset of a straight line. Then there is a non-overlapping optimal solution, i.e., one in which all circles have disjoint interior.*

Proof. An arbitrary optimal solution is modified as follows. For every two overlapping circles C_i and C_{i+1} with centers p_i and p_{i+1}, we decrease r_{i+1}, such that $r_i + r_{i+1} = \text{dist}(p_i, p_{i+1})$, and increase the radius of C_{i+2} by the same amount. This can be iterated, until there is at most one overlap at the outermost circle C_j (with C_{j-1}). Then there must be a point p_{j+1} on the boundary of C_j: otherwise we could shrink C_j contradicting optimality. Decreasing C_j's radius r_j by the overlap l and adding a new circle with radius l around p_{j+1} creates an optimal solution without overlap. □

Theorem 11 *Let P a subset of a straight line g. Then a best 2-circle solution for CRA is a $\frac{5}{4}$-approximation.*

Proof. According to Lemma 10 we are, w.l.o.g., given an optimal solution with non-overlapping circles. Let

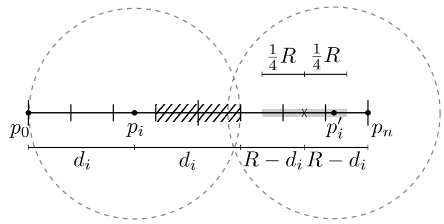

Figure 6: A $\frac{5}{4}$-approximate solution with $d_i < \frac{3}{4}R$. The cross marks the position of the optimal counterpart p_i^* to p_i and the grey area sketches A_i.

p_0 and p_n be the outermost intersection points of the optimal solution circles and g. W.l.o.g., we may further assume $p_0, p_n \in P, R = \sum r_i = \frac{\text{dist}(p_0, p_n)}{2}$ (otherwise, we can add the outermost intersection point of the outermost circle and g to P, which may only improve the approximation ratio). Let p_i denote the rightmost point in P left to the middle of $\overline{p_0 p_n}$ and let p_{i+1} its neighbor on the other half. Further, let $d_i := \text{dist}(p_0, p_i)$, $d_{i+1} := \text{dist}(p_{i+1}, p_n)$ (See Fig. 5). Assume, $d_i \geq d_{i+1}$. We now give $\frac{5}{4}$-approximate solutions using one or two circles that cover $\overline{p_0 p_n}$.

Case 1. If $\frac{3}{4}R \leq d_i$ then $\frac{5}{4}R \geq 2R - d_i = \text{dist}(p_i, p_n)$. Thus, a one-circle solution around p_i is sufficient.

Case 2. If $\frac{3}{4}R > d_i \geq d_{i+1}$ we need two circles to cover $\overline{p_0 p_n}$ with $\frac{5}{4}R$.

Case 2a. The point p_i could be a center point of an optimal two-circle solution if there was a point p_i^* with $\text{dist}(C_i, p_i^*) = \text{dist}(p_i^*, p_n) = R - d_i$. So in case there is a $p'_i \in P$ that lies in a $\frac{1}{4}R$-neighborhood of such an optimal p_i^* we get $\text{dist}(C_i, p'_i), \text{dist}(p'_i, p_n) \leq R - d_i + \frac{1}{4}R$ (see Fig. 6). Thus, $r(p_i) := d_i, r(p'_i) := R - d_i + \frac{1}{4}R$ provides a $\frac{5}{4}$-approximate solution.

Case 2b. Analogously to Case 2a, there is a point $p'_{i+1} \in P$ within a $\frac{1}{4}R$-range of an optimal counterpart to p_{i+1}. Then we can take $r(p_{i+1}) := d_{i+1}$, $r(p'_{i+1}) := R - d_{i+1} + \frac{1}{4}R$ as a $\frac{5}{4}$-approximate solution.

Case 2c. Assume that there is neither such a p'_i nor such a p'_{i+1}. Because d_i, d_{i+1} are in $(\frac{1}{4}R, \frac{3}{4}R)$, we have $\frac{1}{4}R < R - d_j < \frac{3}{4}R$ for $j = i, i+1$, which implies that there are two disjoint areas A_i, A_{i+1}, each with diameter equal to $\frac{1}{2}R$ and excluding all points of P. Because p_i, the rightmost point on the left half of $\overline{p_0 p_n}$, has a greater distance to A_i than to p_0, any circle around a point on the left could only cover parts of both A_i and A_{i+1} if it has a greater radius than its distance to p_0. This contradicts the assumption that p_0 is a leftmost point of a circle in an optimal solution. The same applies to the right-hand side. Thus, $A_i \cup A_{i+1}$ must contain at least one point of P, and therefore one of the previous cases leads to a $\frac{5}{4}$-approximation. □

Fig. 7 shows that the bound is tight. We believe that this is also the worst case when points are *not* on

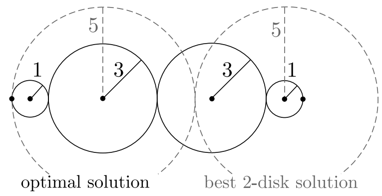

Figure 7: A lower bound of $\frac{5}{4}$ for 2-circle solutions.

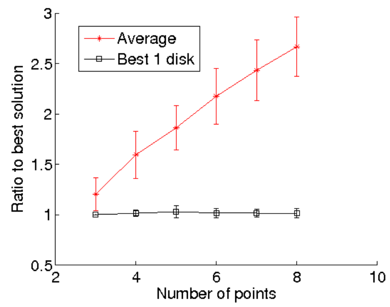

Figure 8: Ratios of the average over all enumerated trees and of the best 1-circle tree to the optimal $\sum r_i$. Results were averaged over 100 trials for each number.

a line. Indeed, the solutions constructed in the proof of Theorem 11 cover a longest path $\mathcal{P}$ in an optimal solution for a general P. If this longest path consists of at most three edges, $p_i(=: p'_{i+1})$ and $p_{i+1}(=: p'_i)$ can be chosen as circle centers, covering all of P. However, if $\mathcal{P}$ consists of at least four edges, a solution for the diameter may produce two internal non-adjacent center points that do not necessarily cover all of P.

5 Experimental Results

It is curious that even in the worst case, a one-circle solution is close to being optimal. This is supported by experimental evidence. For different numbers of uniformly distributed points, we enumerated all possible spanning trees using the method described in [2], and recorded the optimal value with the algorithm mentioned in Section 2. This we compared with the best one-circle solution; as shown in Fig. 8, the latter seems to be an excellent heuristic choice.

6 Conclusion

A number of open problems remain. One of the most puzzling is the issue of complexity in the absence of upper bounds on the radii. The strong performance of the one-circle solution (and even better of solutions with higher, but limited numbers of circles), and the difficulty of constructing solutions for which the one-circle solution is not optimal strongly hint at the possibility of the problem being polynomially solvable. One possible way may be to use methods from linear programming: modeling the objective function and the variables by linear methods is straightforward; describing the connectivity of a spanning tree by linear cut constraints is also well know. However, even though separating over the exponentially many cut constraints is polynomially solvable (and hence optimizing over the resulting polytope), the overall polytope is not necessarily integral. On the other hand, we have been unable to prove NP-hardness without upper bounds on the radii, even in the more controlled context of graph-induced distances.

Other open problems are concerned with the worst-case performance of heuristics using a bounded number of circles. We showed that two circles suffice for a $\frac{4}{3}$-approximation in general, and a $\frac{5}{4}$-approximation on a line; we conjecture that the general performance guarantee can be improved to $\frac{5}{4}$, matching the existing lower bound. Obviously, the same can be studied for k circles, for any fixed k; at this point, the best lower bounds we have are $\frac{7}{6}$ for $k = 3$ and $1 + \frac{1}{2^{k+1}}$ for general k. We also conjecture that the worst-case ratio $f(k)$ of a best k-circle solution approximates the optimal value arbitrarily well for large k, i.e., $\lim_{k\to\infty} f(k) = 1$.

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