The Number of k-Point Subsets Separable by Convex Pseudo-Circles

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Abstract

We show that any set S of n points in the plane contains $2kn-n-k^2+1-\sum_{i=1}^{k-1} a^{(i)}(S)$ subsets of k points that can be separated from the rest of S by convex pseudo-circles (where $a^{(i)}(S)$ denotes the number of i-sets of S). This value does not depend on the set of pseudo-circles.

1 Introduction

Given a finite set S of n points in the plane (no three of them being collinear) and an integer $k \in \{1, ..., n-1\}$, a classical geometric problem consists in searching subsets of k points of S that can be separated from the remaining points by different types of lines. Separation by a straight line has been extensively studied. In this case, the separable subsets are called k-sets. Dey [4] has shown that the number of k-sets of S (denoted by $a^{(k)}(S)$) is bounded by $O(nk^{\frac{1}{3}})$ and Tóth [10] has constructed sets with $n2^{\Omega(\sqrt{\log k})}$ k-sets. In fact, Dey's bound also holds when the separation lines form a family of pseudo-lines, that is x-monotone unbounded curves that pairwise intersect at most once. If the curves are x-monotone and if any two of them may have up to s intersection points (with s even), Buzaglo, Holzman, and Pinchasi [3] have shown that the number of separable k-point subsets is $O(n^{\frac{s}{2}}k^{\frac{s}{2}})$ and that the bound is tight.

When the separation lines are circles and no four points of S are cocircular, Lee [5] proved that S admits $2kn-n-k^2+1-\sum_{i=1}^{k-1} a^{(i)}(S)$ separable k-point subsets (those inside the circles). When the separation lines are pseudo-circles, *i.e.*, simple closed curves that pairwise intersect at most twice, and if all these pseudo-circles either pairwise intersect or enclose a same point of the plane, then the number of separable k-point subsets is $O(nk)$ [3]. The bound is tight.

In this paper we show that Lee's result holds true when the separation lines are convex pseudo-circles. This means that the number of subsets of k points of S that can be separated from the rest of S by convex pseudo-circles is an invariant of S: It does not depend on the set of pseudo-circles.

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2 Order-k Centroid Triangulations

A k-edge is a couple $(P, \{s,t\})$ of subsets of S such that $|P| = k$ and the straight line (st) separates P from $S \setminus (P \cup \{s,t\})$. Relations between k-edges and k-sets can be observed through the so-called k-set polygon of S (denoted by $g^k(S)$), which is the convex hull of the centroids of all k-point subsets of S. Indeed [1]:

Proposition 1 *(i) The centroid $g(T)$ of T is a vertex of $g^k(S)$ if and only if T is a k-set of S.*

(ii) The line segment $g(T)g(T')$ is an edge of $g^k(S)$ if and only if there exists a $(k-1)$-edge $(P, \{s,t\})$ of S such that $T = P \cup \{s\}$ and $T' = P \cup \{t\}$.

In the same way, circularly separable k-point subsets of S correspond to the regions of the order-k Voronoi diagram of S. The edges and vertices of this diagram are characterized by couples (P, Q) of subsets of S such that Q lies on a circle and P is inside the circle: For edges $|Q| = 2$ and $|P| = k-1$, and for vertices $|Q| = 3$ and either $|P| = k-1$ or $|P| = k-2$ (when no four points of S are cocircular) [5]. Taking the centroids of all circularly separable k-point subsets of S and connecting those corresponding to neighbor order-k Voronoi regions, we get a triangulation of the k-set polygon of S called the order-k (centroid) Delaunay triangulation of S [2, 8]. Its edges and triangles are characterized by the above defined couples (P, Q). Here we define the same kind of triangulation, but with convex pseudo-circles instead of circles.

Definition 1 *A couple (P, Q) of disjoint subsets of S is called a (convex) k-couple of S, if there exists a simple closed strictly convex (Jordan) curve γ such that Q lies on γ, P and $S \setminus (P \cup Q)$ lie respectively in the bounded and in the unbounded open region of the plane delimited by γ, and*

- *either $Q = \emptyset$ and $|P| = k$,*
- *or $|P| < k < |P \cup Q|$ and $2 \leq |Q| \leq 3$.*

The second item corresponds to three kinds of k-couples: $|Q| = 2$ and $|P| = k-1$, $|Q| = 3$ and $|P| = k-1$, $|Q| = 3$ and $|P| = k-2$.

The curve γ is said to define the k-couple (P, Q).

Definition 2 *Two distinct k-couples (P, Q) and (P', Q') of S are said to be compatible (with each other) if they admit two defining curves that intersect in at most two points (here a tangent point is considered as a double intersection point).*

Denoting, for every subset E of the plane, $\mathring{E}$ the relative interior of E and $conv(E)$ the convex hull of E, it is easy to see that:

Proposition 2 *(i) (P, Q) is a k-couple of S if and only if P and Q are two disjoint subsets of S such that $conv(P \cup Q) \cap S = P \cup Q$ and*

- *either $Q = \emptyset$ and $|P| = k$,*
- *or the points of Q are extremal points of $P \cup Q$, $|P| < k < |P \cup Q|$, and $2 \leq |Q| \leq 3$.*

(ii) Two distinct k-couples (P, Q) and (P', Q') are compatible if and only if

$$\mathring{conv}((P \cup Q) \setminus P') \cap \mathring{conv}((P' \cup Q') \setminus P) = \emptyset.$$

Definition 3 *For every k-couple (P, Q) of S, let be the set of centroids of all k-point subsets of $P \cup Q$ that contain P. The convex hull of this set of centroids is called the k-set polygon of (P, Q) and is denoted by $g^k(P, Q)$.*

Let us describe the shape of the k-set polygon for every kind of k-couple (P, Q) of Definition 1:

- when $Q = \emptyset$, $g^k(P, Q)$ is the centroid of P,
- when $Q = \{s, t\}$, $g^k(P, Q)$ is the line segment $g(P \cup \{s\})g(P \cup \{t\})$,
- when $Q = \{r, s, t\}$ and $|P| = k - 1$, $g^k(P, Q)$ is the triangle $g(P \cup \{r\})g(P \cup \{s\})g(P \cup \{t\})$; such a triangle is said of type 1,
- when $Q = \{r, s, t\}$ and $|P| = k - 2$, $g^k(P, Q)$ is the triangle $g(P \cup \{r, s\})g(P \cup \{s, t\})g(P \cup \{t, r\})$; such a triangle is said of type 2.

When $k = 1$, the 1-set polygons of the 1-couples of S are the points of S, the segments connecting the points of S, and the triangles with vertices in S and with no point of S inside. Notice that in this case all triangles are type-1. Furthermore, if the 1-couples are pairwise compatible, their corresponding points, open segments and open triangles are pairwise disjoint. The 1-set polygons of any maximal set of compatible 1-couples are then the set of vertices, edges, and faces of a triangulation of S. Conversely, every triangulation of S can be obtained in that way. We extend this to higher values of k.

Using basic centroid properties it is not hard to prove that:

Theorem 3 *If (P, Q) and (P', Q') are two distinct compatible k-couples, $\mathring{g}^k(P, Q)$ and $\mathring{g}^k(P', Q')$ are disjoint.*

The converse of Theorem 3 is generally wrong. It holds notably when $P = P'$ or when $P \cup Q = P' \cup Q'$.

Proposition 4 *The boundary of the k-set polygon of any k-couple (P, Q) with $|Q| \geq 2$ is composed of k-set polygons of k-couples that are compatible with (P, Q), compatible with each others, and compatible with every k-couple that is compatible with (P, Q).*

Observing precisely the boundary of each type of k-set polygon we can see that:

- the endpoints of a segment k-set polygon $g^k(P, \{s, t\})$ are the k-set polygons of the k-couples $(P \cup \{s\}, \emptyset)$ and $(P \cup \{t\}, \emptyset)$,
- the vertices of a type-1 triangle $g^k(P, \{r, s, t\})$ are the k-set polygons of the k-couples $(P \cup \{r\}, \emptyset)$, $(P \cup \{s\}, \emptyset)$, and $(P \cup \{t\}, \emptyset)$. Its edges are the k-set polygons of the k-couples $(P, \{r, s\})$, $(P, \{s, t\})$, and $(P, \{t, r\})$,
- the vertices of a type-2 triangle $g^k(P, \{r, s, t\})$ are the k-set polygons of $(P \cup \{r, s\}, \emptyset)$, $(P \cup \{s, t\}, \emptyset)$, and $(P \cup \{t, r\}, \emptyset)$. Its edges are the k-set polygons of $(P \cup \{r\}, \{s, t\})$, $(P \cup \{s\}, \{t, r\})$ and $(P \cup \{t\}, \{r, s\})$.

Proposition 5 states the same kind of properties for the boundary of the k-set polygon $g^k(S)$ of the whole set S (they can be deduced from Proposition 1).

Proposition 5 *The edges and vertices of the k-set polygon of S are k-set polygons of k-couples that are compatible with each others and with every other k-couple of S.*

The next result is a kind of reciprocal of Proposition 4.

Proposition 6 *If $k \geq 2$ and if $(P_1, \{s_1, t_1\})$, $(P_2, \{s_2, t_2\})$, and $(P_3, \{s_3, t_3\})$ are distinct pairwise compatible k-couples whose k-set polygons form a triangle t then*

(i) t is the k-set polygon of a k-couple,

(ii) this k-couple is compatible with every k-couple that is compatible with $(P_1, \{s_1, t_1\})$, $(P_2, \{s_2, t_2\})$, and $(P_3, \{s_3, t_3\})$.

Proposition 6 is a strong property of k-set polygons of k-couples when $k \geq 2$. Transposed to the case where $k = 1$, it would mean that a triangle with vertices in S cannot contain any point of S inside, which is obviously wrong.

Statement (ii) is a consequence of the following lemma:

Lemma 7 *Let $k \geq 2$ and let $(P, \{r, s, t\})$, $(P_1, \{s_1, t_1\})$, and $(P_2, \emptyset)$ be three compatible k-couples such that $g^k(P_1, \{s_1, t_1\})$ is an edge of $g^k(P, \{r, s, t\})$ and $g(P_2)$ is its opposite vertex.*

If $|P| = k - 1$ (resp. $|P| = k - 2$), every k-couple (P', Q') with $P' \neq P$ (resp. $P' \cup Q' \neq P \cup Q$) that is compatible with $(P_1, \{s_1, t_1\})$ and with $(P_2, \emptyset)$ is also compatible with $(P, \{r, s, t\})$.

This lemma is also the basis of the proof of the following fundamental result:

Theorem 8 *The k-set polygons of any maximal set of distinct compatible k-couples partition the k-set polygon of S in vertices, edges, and triangular faces.*

The triangulation obtained in this way is called an order-k centroid triangulation of S (see Figure 1).

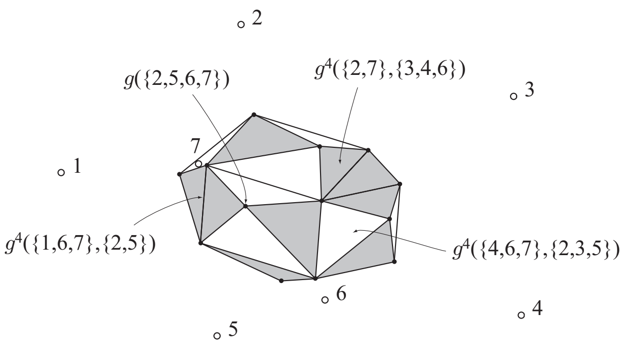

Figure 1: An order-4 centroid triangulation. Type-1 triangles are white and type-2 triangles grey.

An important consequence of Theorem 8, of Propositions 4 and 6, and of Lemma 7 is that:

Corollary 9 *If $\mathcal{C}$ is a maximal set of distinct compatible k-couples then the subset of all k-couples of $\mathcal{C}$ of the form (P, Q) with $Q = \emptyset$ (resp. $|Q| = 2$, resp. $|Q| = 3$ and $|P| = k-1$, resp. $|Q| = 3$ and $|P| = k-2$) is a maximal set of compatible k-couples of S of this form.*

3 Enumeration formulas

Let $\mathcal{T}^k$ be an order-k centroid triangulation of S ($k \geq 2$) and let τ be the set of k-couples (P, Q) whose k-set polygons are the type-2 triangles of $\mathcal{T}^k$ (*i.e.* $|Q| = 3$ and $|P| = k-2$). The k-couples of τ are also pairwise compatible $(k-1)$-couples. The $(k-1)$-set polygons of these $(k-1)$-couples are type-1 triangles and, from Theorem 8, they belong to a same order-$(k-1)$ centroid triangulation $\mathcal{T}^{k-1}$ of S. These triangles are the only type-1 triangles of $\mathcal{T}^{k-1}$: Otherwise there would exist a $(k-1)$-couple $(P', Q') \notin \tau$ with $|P'| = k-2$ and $|Q'| = 3$ compatible with the elements of τ. Since (P', Q') is also a k-couple compatible with the elements of τ, this would be in contradiction with Corollary 9. Hence:

Lemma 10 *The number of type-2 triangles of $\mathcal{T}^k$ is equal to the number of type-1 triangles of $\mathcal{T}^{k-1}$.*

There is also a relation between the vertices of $\mathcal{T}^k$ and the edges of $\mathcal{T}^{k-1}$.

Lemma 11 *$g(T)$ is a vertex of $\mathcal{T}^k$ if and only if there exists an edge $g^{k-1}(P, \{s, t\})$ in $\mathcal{T}^{k-1}$ with $P \cup \{s, t\} = T$.*

Proof. One can show that every vertex $g(T)$ of $\mathcal{T}^k$ is adjacent to at least one type-2 triangle. This triangle is then of the form $g^k(P, \{r, s, t\})$ with $T = P \cup \{s, t\}$. Thus, $g^{k-1}(P, \{r, s, t\})$ is a type-1 triangle of $\mathcal{T}^{k-1}$ and $g^{k-1}(P, \{s, t\})$ is one of its edges.

Conversely, if $g^{k-1}(P', \{s', t'\})$ is an edge of $\mathcal{T}^{k-1}$ with $(P', \{s', t'\}) \neq (P, \{s, t\})$, it is easy to see that $(P' \cup \{s', t'\}, \emptyset)$ is a k-couple compatible with $(P \cup \{s, t\}, \emptyset) = (T, \emptyset)$ and, from Corollary 9, $g(P' \cup \{s', t'\})$ is a vertex of $\mathcal{T}^k$. ☐

If, for a k-point subset T of S, the union of the edges and triangles of $\mathcal{T}^{k-1}$ of the form $g^k(P, Q)$ with $P \cup Q = T$ is not empty, then this union is called the domain of T in $\mathcal{T}^{k-1}$.

Clearly, every edge and every type-2 triangle of $\mathcal{T}^{k-1}$ belongs to one and only one domain. The edges of a type-2 triangle belong to the same domain as the triangle. The type-1 triangles belong to no domain. From Lemma 11, we then have:

Lemma 12 *The number of vertices of $\mathcal{T}^k$ is equal to the number of domains of $\mathcal{T}^{k-1}$.*

The right knowledge of the domains' shape will be needed to enumerate the vertices of $\mathcal{T}^k$.

Proposition 13 *(i) The vertices of $\mathcal{T}^{k-1}$ that belong to a same domain are the extremal points of this domain.*

(ii) Every domain is convex.

Proof. (i) Every vertex in the domain of T is of the form $g(T \setminus \{s\})$ since it is an endpoint of an edge of the form $g^{k-1}(T \setminus \{s, t\}, \{s, t\})$. Therefore, s is an extremal point of T and, by an homothety centered in $g(T)$, $g(T \setminus \{s\})$ is an extremal point of the domain.

(ii) We show that the line segment connecting any two vertices $g(T \setminus \{r\})$ and $g(T \setminus \{s\})$ of the domain of T also belongs to this domain. Let $g^{k-1}(T \setminus \{s, t\}, \{s, t\})$ be an edge of the domain of T with endpoint $g(T \setminus \{s\})$. If $t = r$ we are done. Otherwise, since $conv(T) \cap S = T$, since $|T| = k$, and since r, s, t are extremal points of T, $(T \setminus \{r, s, t\}, \{r, s, t\})$ is a $(k-1)$-couple of S. Now, $g^{k-1}(T \setminus \{s, t\}, \{s, t\})$ and $g(T \setminus \{r\})$ are respectively an edge and a vertex of the type-2 triangle $g^{k-1}(T \setminus \{r, s, t\}, \{r, s, t\})$. From Lemma 7, Proposition 4, and Theorem 3, only $(k-1)$-set polygons $g^k(P, Q)$ with $P \cup Q = T$ may cut the edge $g(T \setminus \{r\})g(T \setminus \{s\})$ of $g^{k-1}(T \setminus \{r, s, t\}, \{r, s, t\})$. Hence, this edge belongs to the domain of T. ☐

Theorem 14 *The number of vertices of any order-k centroid triangulation of S ($k \in \{1, \ldots, n-1\}$) is*

$$2kn - n - k^2 + 1 - \sum_{i=1}^{k-1} a^{(i)}(S).$$

Proof. Consider a sequence $(\mathcal{T}^1, \ldots, \mathcal{T}^k)$ of order-1, $\ldots$, order-k centroid triangulations of S such that, for all $i \in \{2, \ldots, k\}$, the type-1 triangles of $\mathcal{T}^{i-1}$ correspond to the type-2 triangles of $\mathcal{T}^i$. Set $v^{(k)}, e^{(k)}, t_1^{(k)}$,

$t_2^{(k)}$ the respective numbers of vertices, edges, type-1 and type-2 triangles of $\mathcal{T}^k$. Since $\mathcal{T}^k$ is a triangulation of $g^k(S)$ and since the number of vertices of $g^k(S)$ equals the number $a^{(k)}(S)$ of k-sets of S, we get

$$e^{(k)} = 3v^{(k)} - a^{(k)}(S) - 3, \qquad (1)$$

$$t_1^{(k)} + t_2^{(k)} = 2v^{(k)} - a^{(k)}(S) - 2. \qquad (2)$$

Denoting by $e_T^{(k-1)}$ and $t_T^{(k-1)}$ the numbers of edges and of triangles of the domain of T in $\mathcal{T}^{k-1}$, Proposition 13 implies

$$e_T^{(k-1)} = 2t_T^{(k-1)} + 1.$$

Since every edge and every type-2 triangle of $\mathcal{T}^{k-1}$ belongs to one and only one domain, it then follows from Lemma 12,

$$e^{(k-1)} = 2t_2^{(k-1)} + v^{(k)}.$$

Thus, from Lemma 10,

$$e^{(k-1)} + e^{(k-2)} = 2(t_1^{(k-2)} + t_2^{(k-2)}) + v^{(k)} + v^{(k-1)}$$

and, using equations (1) and (2),

$$v^{(k)} - 2v^{(k-1)} + v^{(k-2)} = a^{(k-2)}(S) - a^{(k-1)}(S) - 2.$$

The result follows by induction on k and by the fact that $v^{(1)} = n$ (the vertices of $\mathcal{T}^1$ are the points of S) and that $v^{(2)} = 3n - a^{(1)}(S) - 3$ (the domains of $\mathcal{T}^1$ are the edges of $\mathcal{T}^1$). □

4 Discussion

Theorems 8 and 14 show that the number of subsets of k points of S that can be separated from the rest of S by convex pseudo-circles is equal to $2kn - n - k^2 + 1 - \sum_{i=1}^{k-1} a^{(i)}(S)$, and is an invariant of S. Furthermore, the couples (P, Q) such that P is a k-point subset separable by a convex pseudo-circle passing through Q correspond to the edges (when $|Q| = 2$) and to the type-1 triangles (when $|Q| = 3$) of an order-$(k+1)$ centroid triangulation. Their numbers can be deduced from proof of Theorem 14.

The convex pseudo-circles considered in this paper are closed curves. The results also hold for curves closing at infinity (as parabolas) if we consider the separable subset as the one belonging to the convex region of the plane delimited by the curve.

The separable subsets treated in this paper are closely related to the following ones: Let $\mathcal{F}$ be a family of subsets of S such that each member of $\mathcal{F}$ is the intersection of S with a convex set and, for any two members A and B of $\mathcal{F}$, $A \not\subseteq B$, $B \not\subseteq A$, and both $conv(A) \setminus conv(B)$ and $conv(B) \setminus conv(A)$ are connected (or empty). Pinchasi and Rote [7] proved that $\mathcal{F}$ contains at most $4\binom{n}{2} + 1$ members and that the bound is tight (within a constant multiplicative factor). Notice however that the constraints on the convex hulls in this definition are stronger than our compatibility condition.

In [5], Lee proposed an algorithm to construct the order-k Voronoi diagram by deducing it from the order-$(k-1)$ Voronoi diagram. This algorithm can be dualized to iteratively construct the order-k centroid Delaunay triangulation, starting with the classical (order-1) Delaunay triangulation [8]. In order to generate suitable bases for multivariate B-spline spaces, Liu and Snoeyink [6] proposed to extend this algorithm, starting with any (order-1) triangulation. When their algorithm succeeds in constructing a triangulation, then this triangulation is called a centroid triangulation. They proved that their algorithm actually works for $k \leq 3$ and conjectured that it works for all k. We can prove that our order-k centroid triangulations can be generated by such an algorithm and, conversely, that every execution of the algorithm generates such a triangulation [9]. This proves Liu and Snoeyink's conjecture.

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