Boundary of a non-uniform point cloud for reconstruction

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Abstract

This paper deals with the problem of shape reconstruction. We define a filtration of the Delaunay complex of a point cloud. This filtration allows to select points in a point cloud that should be boundary points. Theoretical guarantees are given when the point cloud samples a region with smooth boundary. A simple and efficient algorithm computing the filtration is described.

1 Introduction

We will focus on the reconstruction of an open set Ω in $\mathbb{R}^d$ with smooth boundary, from a set of points non-uniformly distributed inside Ω and not just lying on its boundary.

With our hypothesis, $\overline{\Omega}$, the closure of Ω, is a manifold with boundary. In the past decade, many algorithms have been proposed for the reconstruction of manifolds [2, 3, 4]. Some of them give guarantees, but only for surfaces without boundary. Recently, in [7], Dey et al. have been the firsts to ensure the reconstruction of surfaces with boundary. Nevertheless, their algorithm is limited to 2-dimensional surfaces in $\mathbb{R}^3$, from a sufficiently dense sample. Besides, their algorithm faces problems when the sample is not uniform and no theoretical guarantees are provided in this case.

Given a point cloud S sampling a bounded open set Ω in $\mathbb{R}^d$ with smooth boundary, our work proposes:

- To define a filtration of the Delaunay complex of S. This filtration takes into account in a very natural way that the local density of the cloud may change according to the local complexity of Ω. This filtration depending on a positive real number α is called *Locally-Density-Adaptative-α-complex, LDA-α-complex* for short.
- To select a subset in the point cloud S, called *LDA-α-boundary*, that presumably constitutes a sample of the boundary F of Ω.
- Thanks to a result of Chazal and Lieutier [6] to prove that the LDA-α-boundary carries topological and geometric properties of F.
- To give an efficient algorithm for computing LDA-α-complex and LDA-α-boundary.

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2 Mathematical and geometric concepts

Notations. $d(x,y) = \|x-y\|$ denotes the Euclidean distance between to points x, y in $\mathbb{R}^d$.

$B(x,r) = \{y \in \mathbb{R}^d, d(x,y) \leq r\}$ denotes the closed ball of center x and radius $r \in [0,\infty[$.

Let A be a subset of $\mathbb{R}^d$, A^C denotes its complementary in $\mathbb{R}^d$, $\overline{A}$ its closure, A^o its interior, ∂A its boundary, $\mathrm{conv}(A)$ its convex hull, and $\mathrm{diam}\, A$ its diameter, i.e. $\mathrm{diam}\, A = \sup\{d(x,y) : x, y \in A\}$.

Delaunay complex. Let S be a finite point set in $\mathbb{R}^d$. An *empty ball* is a ball containing no point of S in its interior.

The *Delaunay complex* of S, $\mathrm{Del}(S)$, is the set of convex polytopes $K = \mathrm{conv}(S \cap B)$ where B is an empty ball.

Maximal ball. A ball B of $\mathbb{R}^d$ is called a *maximal ball* if B is an empty ball such that the dimension of the convex polytope $\mathrm{conv}(B \cap S)$ is d.

Restricted Delaunay complex. Let X be a point set in $\mathbb{R}^d$. A convex polytope K in the Delaunay complex $\mathrm{Del}(S)$ is in the Delaunay complex of S restricted to X, denoted by $Del_{|X}(S)$, if there exists an empty ball $B = B(x,r)$ such that $x \in X$ and the vertices of K are on the boundary of B. This means that K is a face of $\mathrm{conv}(S \cap B)$. The set $Del_{|X}(S)$ of all these convex polytopes K is a sub-complex of $\mathrm{Del}(S)$.

We assume in the following that S is a finite point set in $\mathbb{R}^d$ which is contained in no proper affine subspace of $\mathbb{R}^d$.

3 Idea of sampling and reconstruction

We aim at reconstructing a bounded open set Ω with smooth boundary from a sample set S included in $\overline{\Omega}$ and not only on its boundary. Moreover, the sample does not have to be uniformly distributed, i.e., its local density may be different in places. Nevertheless, the information about the shape to reconstruct has to be carried by the sample. The sample has to be dense where a big amount of details is required and a sudden variation of local density indicates the presence of a hole or a hollow, as shown on figure 1 close to the eyes of the lizard or on its fingers. Since variations of the local density are allowed, the sample may be less dense where the shape is simple. However, the density must change gradually, as shown on figure 1 on the back of the lizard, to avoid the formation of non-existent holes.

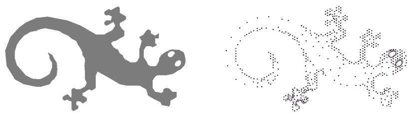
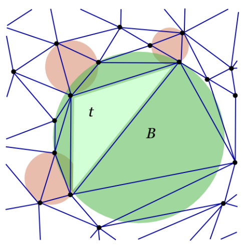


Figure 1: A shape (left), a possible sample (right)

The problem comes down to finding an efficient way to measure the variation of local density of the sample set S.

4 Formalization of the idea

The Delaunay complex can be efficiently used to measure the density variation of the sample set S. For instance, in the figure opposite, the maximal ball B is much larger than some of its neighbours. This means there is a wide area with no point of S while the sampling around is dense. This area is probably a hole and the polytope t included in B must be eliminated.



4.1 Eliminating polytopes

Polytopes will be eliminated by eliminator balls. An eliminator ball is an empty ball large enough compared to the maximal balls around.

Definition 1 *Let α be a real number in $]0, \infty[$. An empty ball $B = B(c, r)$ is* α-eliminator *if for each point p of S lying on ∂B, there exists a maximal ball $B' = B'(c', r')$ with p lying on $\partial B'$ such that $r \geq \alpha r'$.*

Here are some interesting observations:

- There exists an α_∞ such that $\forall \alpha \geq \alpha_\infty$ there is no α-eliminator ball since the radius of a maximal ball is strictly positive and the radius of an empty ball is finite.

- There exists an α_0 such that $\forall \alpha \leq \alpha_0$ all empty balls containing a 1-polytope are α-eliminator since the radius of an empty ball containing a 1-polytope is strictly positive and the radius of a maximal ball is finite.

- A maximal ball $B = B(c, r)$ is 1-eliminator since the ball $B'(c', r')$ with p lying on $\partial B'$ such that $r \geq r'$ is B itself.

Definition 2 *A polytope K of* Del(S) *is* α-eliminated *if each empty ball B such that $K \subseteq$* conv$(S \cap \partial B)$ *is α-eliminator.*

The following observations are deduced from the previous ones: No polytope is α_∞-eliminated. All the polytopes except the 0-polytopes are α_0-eliminated. All d-polytopes are 1-eliminated.

4.2 LDA-α-complex

Definition 3 *The* LDA-α-complex *is the set of all polytopes K of* Del(S) *that are not* α-eliminated.

We can collect some easy facts: The LDA-α-complex of S is a sub-complex of Del(S). The LDA-α_0-complex of S is S. The LDA-α_∞-complex of S is Del(S). The LDA-1-complex of S is a sub-complex of Del(S) with no d-polytope. If $\alpha_1 \leq \alpha_2$ then LDA-α_1-complex $\subseteq$ LDA-α_2-complex.

The LDA-α-complex is indeed very close to the conformal-α-shape [5] with α_p^- equals to 0 and α_p^+ equals to the radius of the smallest maximal ball containing p.

4.3 LDA-α-boundary

Definition 4 *The* LDA-α-boundary *is the union of the sets of the vertices of all the α-eliminated polytopes and of the set of all the vertices of the convex hull of S.*

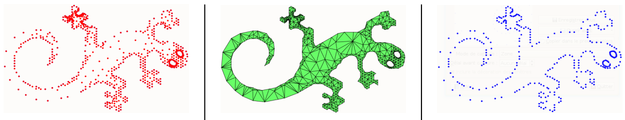
5 Algorithm

We present a very simple and efficient algorithm in the full version of this paper. A short description is given in what follows.

5.1 Description

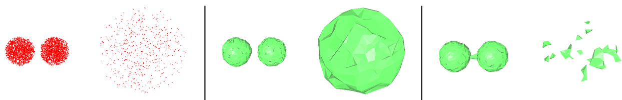
There are four main steps. First, Del(S) is computed. The second step computes for each $v \in S$, the radius $m(v)$ of the smallest maximal ball containing v .The third step computes for each polytope K in Del(S), the radius $e(K)$ of the smallest empty ball containing K. The fourth step determines the polytopes in Del(S) that are in the LDA-α-complex.

The fourth step selects the polytopes of Del(S) in decreasing order of dimensions. First, for each d-polytope K, if the inequality $e(K) < \alpha m(v)$ holds for at least one vertex v of K, then K is in the LDA-α-complex. Moreover, the LDA-α-boundary is determined at the same time since the elements of the LDA-α-boundary are the vertices of the d-polytopes that are not selected (together with the vertices of the convex hull of S). Next, once the polytopes of dimensions $\geq k$ have been selected, a $(k-1)$-polytope K of Del(S) is in the LDA-α-complex if either one of the



A sample of a lizard | Its LDA-2 -complexe | Its LDA-2 -boundary

Figure 2: The lizard



Three balls sampled (a) | with LDA-α-shapes (b) | with α-shapes (c)

Figure 3: The balls

k-polytope containing K is in the LDA-α-complex or if $e(K) < \alpha m(v)$ for at least one vertex v of K.

In the full version of this paper we prove that the time-complexity of the algorithm is $O(|\mathrm{Del}(S)|)$ plus the time required to compute the Delaunay complex $\mathrm{Del}(S)$.

5.2 Little examples

Thanks to CGAL [1], this algorithm has been implemented in 2D and 3D. Two examples are shown in this subsection. Figure 2 shows the sample of a lizard, its LDA-2-complex, and its LDA-2-boundary. The density of this sample is strong only where it is needed. Figure 3 (a) shows 3D-balls that have been randomly sampled according to non-constant probability densities. The reconstructions work with LDA-α-complexes while it is well known that α-shapes may fail in that case.

6 Well distributed subsets

Given a bounded open set Ω with smooth boundary and a finite point set S in $\overline{\Omega}$, we will give some conditions about S ensuring that with a right choice of α, the LDA-α-boundary of S carries the topological informations about $\partial\Omega$.

We first define (ε, δ)-sample with a general lipschitz function. We will specify such a function later.

Definition 5 *Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be a k-lipschitz function. Let δ and ε be two positive real numbers.*

A point set S is an (ε, δ)-sample of $\overline{\Omega}$ if $S \subset \overline{\Omega}$, and if

Density: $\forall x \in \overline{\Omega},\ \exists y \in S,\ d(x, y) < \varepsilon f(x)$,
Sparsity: $\forall x, y \in S,\ x \neq y \Rightarrow d(x, y) \geq \delta f(x)$.

In the two following lemmas, we assume that $\overline{\Omega}$ is (ε, δ)-sampled and that f is k-lipschitz.

7 Two technical lemmas

We want to provide theoretical guarantees that the LDA-α-boundary is a good approximation of the boundary F of Ω.

To do this we have to prove these two results: (1) Any point in the LDA-α-boundary is near F. (2) Any point in F is near a point of the LDA-α-boundary.

Next lemma shows that with a right choice of the parameters $\alpha, \varepsilon, \delta$, the LDA-$\alpha$-boundary is close to F.

Lemma 6 ***(elimination lemma).*** *Suppose $\varepsilon < \frac{1}{2k}$ and $\alpha > \frac{2\varepsilon}{\delta(1-2k\varepsilon)}$. If p is a vertex of a LDA-α-eliminated polytope t_p, then*

$$d(p, F) \leq \frac{\varepsilon}{1 - k\varepsilon} f(p) \qquad (1)$$

$$d(c, \overline{\Omega}) > 0 \qquad (2)$$

where c is the center of any empty ball B such that $t_p \subset \mathrm{conv}(S \cap B)$.

7.1 Statements of main results about (ε, δ)-samples

The first inequality in the elimination lemma shows immediately that points in the LDA-α-boundary are close to F:

Corollary 7 *(α-**boundary**). Suppose $\alpha > \frac{2\varepsilon}{\delta(1-2k\varepsilon)}$ and $\varepsilon < \frac{1}{2k}$. Then all $p \in S$ that are in the LDA-α-boundary of S, are at a distance from F lower or equal to $\frac{\varepsilon}{1-k\varepsilon} f(p)$.*

The second inequality in the elimination lemma immediately leads to an information about $\mathrm{Del}_{|\overline{\Omega}}(S)$ the restricted Delaunay triangulation of S with respect to $\overline{\Omega}$:

Corollary 8 *Suppose $\alpha > \frac{2\varepsilon}{\delta(1-2k\varepsilon)}$ and $\varepsilon < \frac{1}{2k}$. Then the restricted Delaunay complex $\mathrm{Del}_{|\overline{\Omega}}(S)$ is a subcomplex of the LDA-α-complex.*

Until now, we only assume the function f to be k-lipschitz. In order to go further, we need to specify the function f. Let $\mathcal{S}_F$ be the *skeleton* of F, that is the set of centers of maximal open balls that do not meet F. In the following $f : \mathbb{R}^d \to \mathbb{R}$ is defined by $f(x) = d(x, F) + d(x, \mathcal{S}_F)$. It is a 2-lipschitz function.

With this more precise assumption about the sample set S, we are not able to prove the reverse inclusion LDA-α-complex $\subset \mathrm{Del}_{|\overline{\Omega}}(S)$. Nevertheless, we can prove that the LDA-α-complex is included in the Delaunay complex of S restricted to a neighborhood of $\overline{\Omega}$.

Proposition 9 ***(neighborhood).*** *Suppose that $\varepsilon < \frac{1}{20}$ and $\frac{1}{4\varepsilon} > \alpha > 2$. Then a maximal ball $B(c, R)$ such that $d(c, \overline{\Omega}) \geq \frac{2\varepsilon\alpha}{1-4\varepsilon\alpha} f(c)$ is α-eliminator.*

By the first corollary, any point in the LDA-α-boundary is close to F. Conversely any point in F is close to a point of the LDA-α-boundary of S:

Proposition 10 (*α-boundary*). *Suppose that $\varepsilon < \frac{1}{20}$ and $\frac{1}{32\varepsilon} > \alpha > 2$. Then for any p in F there exists a point p_0 in the LDA-α-boundary of S whose distance to p is $\leq 8\alpha\varepsilon f(p)$.*

8 A reconstruction result

The last step is to see that with a good choice of α, ε, and δ, the topological information about F is contained in the α-boundary. For this, we will use a result of F. Chazal and A. Lieutier [6]. Their result allows a reconstruction of a compact manifold Σ of dimension $d-1$ starting with a compact set K close to Σ. They have proved that if K is close enough to Σ, then the boundary of a union of balls $B(p, r_p)$ centered at the points p in K, is the union of two subsets homeomorphic to Σ. This shows that K carries the topological informations about Σ.

Notations. Let Σ be a compact set in $\mathbb{R}^d$. For x in $\mathbb{R}^d$, denote by $\pi(x)$ a point in Σ nearest to x. This point is unique if x is not in the medial axis of Σ. Denote by $LFS(x)$ the distance from x to the skeleton of Σ.

Definition 11 *Let κ and ρ be positive real numbers, and let Σ be a compact manifold in $\mathbb{R}^d$. A compact set $K \subset \mathbb{R}^d$ is a (κ, ρ)-approximation of Σ if :*
i. For all $p \in K$, $d(p, \pi(p)) \leq \kappa\rho LFS(p)$,
ii. For all $p \in \Sigma$, there exists a point $q \in K$ such that $d(p, \pi(q)) < \rho LFS(p)$.

Theorem 12 *(Chazal, Lieutier) Let κ, ρ, and $0 < a < b < \frac{1}{3} - \kappa\rho$ be such that*

$$(1-a')^2 + \left((b'-a') + \frac{b(1+2b'-a')}{1-b-\kappa\rho}\right)^2 < \left(1 - \frac{\kappa\rho(1+2b'-a')}{1-b-\rho}\right)^2$$

with $a' = (a-\kappa\rho)(1-\rho) - \rho$ and $b' = \frac{b+\kappa\rho}{1-2(b+\kappa\rho)}$. Let K be a (κ, ρ)-approximation of Σ. Let $(r_p)_{p\in K}$ be a family of real numbers such that $a \leq \frac{r_p}{LFS(p)} \leq b$ for all $p \in K$ and set

$$\mathcal{K} = \mathcal{K}((r_p)_{p\in K}) = \bigcup_{p\in K} B(p, r_p).$$

Then
- Σ is a deformation retract of $\mathcal{K}$.
- $\mathcal{K}$ is homeomorphic to any tubular neighborhood $\{x \in \mathbf{R}^d : d(x, \Sigma) \leq s\}$ where $s < reach(\Sigma) = d(\Sigma, \mathcal{M}_\Sigma)$.
- The boundary $\partial\mathcal{K}$ is an isotopic hypersurface to $\Sigma_s = \{x \in \mathbb{R}^d : d(x, \Sigma) = s\}$.

Remark 1. If Σ is an hypersurface, $\partial\mathcal{K}$ is isotopic to 2 copies of Σ (isotopy$\Longrightarrow$ homeorphism).

In order to reconstruct the boundary $F = \partial\Omega$ starting with the (ε, δ)-sample set S, it is enough to use this theorem with $\Sigma = F$ and K the LDA-α-boundary of S.

Proposition 13 *Choose $\alpha = 10$, $\varepsilon \leq \frac{1}{2500}$, and $\delta = \frac{\varepsilon}{4}$. Then Chazal and Lieutier's theorem works with $\Sigma = F$ and $K =$ the LDA-α-boundary of S*

All the proofs are in the full version of this paper.

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