4-Holes in Point Sets

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Abstract

We consider a variant of a question of Erdős on the number of empty k-gons (k-holes) in a set of n points in the plane, where we allow the k-gons to be non-convex. We show bounds and structural results on maximizing and minimizing the number of general 4-holes, and maximizing the number of non-convex 4-holes.

1 Introduction

Let S be a set of n points in general position in the plane. A k-gon is a simple polygon spanned by k points of S. A *k-hole* is an empty k-gon, that is, a k-gon which does not contain any points of S in its interior.

In 1978 Erdős [5] raised the following question for convex k-holes: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?" As already observed by Esther Klein, every set of 5 points determines a convex 4-hole, and 10 points always contain a convex 5-hole, a fact proved by Harborth [8]. However, in 1983 Horton showed that there exist arbitrarily large sets of points not containing any convex 7-hole [9]. It again took almost a quarter of a century after Horton's construction to answer the existence question for 6-holes. In 2007/08 Nicolás [11] and independently Gerken [7] proved that every sufficiently large point set contains a convex 6-hole.

Erdős also proposed the following variation of the problem [6]. "What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?" We know by Horton's construction that $h_k(n) = 0$ for $k \geq 7$. For $k \leq 6$, upper and lower bounds on $h_k(n)$ exist; see [1] for a survey.

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In this paper we generalize the latter problem by allowing a k-hole to be non-convex. Thus, whenever we refer to a k-hole, it might be convex or non-convex, and we will explicitly state it when we restrict investigations to one of these two classes.

Note that a set of four points in non-convex position might span up to three 4-holes; that is, the number of k-holes can be larger than $\binom{n}{k}$, the maximum number of convex k-holes.

We first investigate sets of small cardinality (Section 2), and then consider the following tasks: maximizing the number of 4-holes (Section 3), maximizing the number of non-convex 4-holes (Section 4), and minimizing the number of 4-holes (Section 5). In addition to the best possible lower and upper bounds on their number, we also show which families of point sets obtain these bounds.

2 Small Sets

Even to determine the number of small holes is surprisingly intriguing. For $n \leq 11$, Table 1 shows the minimum number of convex 4-holes, the maximum number of non-convex 4-holes, the minimum and maximum number of (general) 4-holes, and, for easy comparison, the number of 4-tuples.

n	convex min	non-convex max	general min	general max	$\binom{n}{4}$
4	0	3	1	3	1
5	1	8	5	9	5
6	3	18	15	22	15
7	6	36	35	43	35
8	10	64	66	77	70
9	15	100	102	126	126
10	23	150	147	210	210
11	32	216	203	330	330

Table 1: Number of 4-holes for $n = 4, \ldots, 11$ points [1].

Obviously, the maximum number of convex 4-holes is $\binom{n}{4}$, obtained by sets in convex position. For minimizing the number of convex 4-holes, the currently best known bounds are $\frac{n^2}{2} - O(n) \leq h_4(n) \leq 1.9397n^2 + o(n^2)$, see [3, 4, 12].

For $n = 4, \ldots, 7$ it can be seen from Table 1 that the minimum number of 4-holes is $\binom{n}{4}$. In contrast, $\binom{n}{4}$ is the maximum number of 4-holes for $n = 9, \ldots, 11$, so the structure of extremal sets seems to switch.

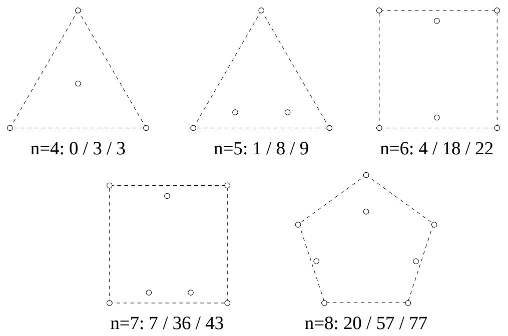

Figure 1: Point sets maximizing the number of 4-holes for $n = 4, \ldots, 8$. Shown are the number of convex, non-convex, and general 4-holes.

Figure 1 shows point sets maximizing the number of 4-holes for $n = 4, \ldots, 8$. The results for $n > 8$ suggest that sets in convex position might maximize the number of 4-holes for $n \geq 9$. Indeed, this will be the first result we prove for general 4-holes (Section 3).

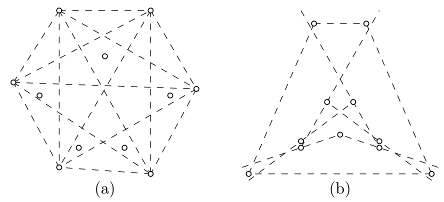

Figure 2: Two unique extremal sets for $n = 11$ points: (a) maximizes the number of non-convex 4-holes, and (b) minimizes the number of 4-holes.

Figure 2 shows two extremal sets for $n = 11$ points. Each set represents the unique order type which reaches the extreme value. The left set maximizes the number of non-convex 4-holes, namely 216, and consists of a convex 5-hole inside a convex 6-gon. The total number of 4-holes in this set is 267; i.e., it contains in addition 51 convex 4-holes. The set on the right side minimizes the number of general 4-holes. It contains 51 convex and 152 non-convex 4-holes, thus in total the minimum of 203 4-holes.

3 Maximizing the Number of (General) 4-Holes

Lemma 1 *Let Δ be a non-empty triangle in S. There are at most three non-convex 4-holes spanned by the three vertices of Δ plus a point of S in the interior of Δ.*

Proof. Let p_1, p_2, and p_3 be the vertices of Δ. Observe that any non-convex 4-hole has to use two edges of Δ. Thus there are three choices for the unused edge of Δ, and for each choice there is at most one way to complete the two used edges of Δ to a 4-hole. Assume to the contrary that two different 4-holes avoid the edge p_2p_3 and use points q_1 and q_2, respectively, in the interior. Then q_2 lies outside the two triangles formed by $p_1q_1p_2$ and $p_1q_1p_3$. Thus q_2 lies in the triangle formed by $p_2q_1p_3$. But then q_1 must lie in one of the triangles spanned by $p_1q_2p_2$ and $p_1q_2p_3$, a contradiction. □

Theorem 2 *For $n \geq 9$ the number of 4-holes is maximized by a set of n points in convex position.*

Proof. In the following we assign every non-convex 4-tuple of points to the three vertices of its convex hull and call this the *representing* triangle of the potential non-convex 4-holes. By Lemma 1, any non-empty triangle represents at most three 4-holes, and any convex 4-tuple gives at most one 4-hole.

Let T be the number of non-empty triangles. As any non-empty triangle induces at least one 4-tuple in non-convex position, we get

$$\binom{n}{4} + 2T \tag{1}$$

as a first upper bound on the number of 4-holes of a point set.

Note that a triangle Δ with $k \geq 1$ interior points is counted $k+2$ times in (1), namely k times in the $\binom{n}{4}$ 4-tuples plus the extra 2 as Δ is non-empty. Thus for $k > 1$ we have over-counted the number of non-convex 4-holes assigned to Δ; cf. Lemma 1. Moreover, many of the convex 4-gons might not be empty and thus no 4-holes. Therefore we now analyze how many of the counted potential 4-holes can be reduced from (1). We will do this by assigning (possibly multiple) markers for over-counted 4-holes to convex 4-tuples and non-empty triangles.

As above, let Δ be a triangle with $k \geq 1$ interior points, and consider all 4-tuples consisting of the three vertices of Δ plus an extra point p. We distinguish two cases.

First let p be one of the n–k–3 points outside Δ. If the resulting 4-tuple is convex, we mark this 4-tuple, as it is not empty and thus no 4-hole. If the 4-tuple is non-convex, we mark the triangle which represents the potential non-convex 4-hole, as at least one of the three possible 4-holes of this 4-tuple is non-empty.

In the second case we consider the k points inside Δ. As argued above, Δ was counted $k+2$ times. But by Lemma 1, there are at most three 4-holes using one interior point of Δ and thus represented by Δ. Therefore we assign $k-1$ markers to Δ.

Altogether we have distributed $n-4$ markers while considering Δ. Repeating this for all non-empty triangles, we obtain a total of $(n-4) \cdot T$ markers.

A non-empty convex 4-tuple might have received up to 4 markers in this way, one from each of its subtriangles. That is, we have at most 4 times as many markers as convex 4-tuples which we can reduce from the upper bound (1).

A non-empty triangle Δ with $k \geq 1$ interior points might have got $4 \cdot (k-1)$ markers: For its interior points, Δ received $k-1$ markers from the second case, and for each non-empty triangle formed by two vertices of Δ and one point inside Δ, we received one marker from the first case. As at least three of the considered inner triangles are empty (the ones spanned by an edge e of Δ and the interior point closest to e), the first case gives at most $3 \cdot (k-1)$ additional markers, resulting in a total of at most $4 \cdot (k-1)$ markers for Δ. As Δ was counted $k+2$ times, but represents at most three 4-holes (Lemma 1), we have at most $4 \cdot (k-1)$ markers for at least $(k+2)-3 = k-1$ over-counted objects. Thus, in both cases we over-counted reducible terms at most by a factor of 4. We therefore can reduce the number of potential 4-holes by one quarter of the distributed markers, namely by $\frac{n-4}{4} \cdot T$. This leads to the improved upper bound

$$\binom{n}{4} + 2T - \frac{n-4}{4} \cdot T$$

for the number of 4-holes. For $n \geq 12$ this is at most $\binom{n}{4}$, the number of 4-holes for a set of points in convex position. Together with the results from Table 1 for $n = 9, \ldots, 11$, this proves the theorem. ☐

4 Maximizing the Number of Non-Convex 4-Holes

Lemma 3 *The number of non-convex 4-holes of any set of n points is at most $\frac{n(n-1)(n-2)}{2} = \frac{n^3}{2} - O(n^2)$.*

Proof. By Lemma 1, any non-empty triangle generates at most three non-convex 4-holes, and there are at most $\binom{n}{3}$ such triangles in a set of n points. ☐

Theorem 4 *For every $n' > 0$ there exist sets of n points for some $n \in \{n', \ldots, 2n'\}$, with at least $\frac{n^3}{2} - O(n^2 \log n)$ non-convex 4-holes.*

Proof. We consider the special point sets $\mathcal{X}_m$, $m \geq 1$, with $|\mathcal{X}_m| = n = 2^{m+1}-2$ points, introduced in [10]. The point sets are defined recursively in layers, starting with two points $\mathcal{X}_1 := \mathcal{R}_1$ in the first layer. An additional layer $\mathcal{R}_i$ is added to $\mathcal{X}_{i-1} := \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{i-1}$ by placing two new points close to any point in $\mathcal{R}_{i-1}$ outside the convex hull of $\mathcal{X}_{i-1}$, such that the following conditions hold: (1) $\mathcal{X}_i = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_i$ is in general position, (2) $\mathcal{R}_i$ are the extremal points of $\mathcal{X}_i$, and (3) any triangle determined by $\mathcal{R}_i$ contains precisely one point of $\mathcal{X}_i$ in its interior. See Figure 3 for an example and [10] for a detailed description of the construction. Furthermore, in [10] it is shown that every triangle spanned by $\mathcal{X}_m$ contains at most one interior point of $\mathcal{X}_m$; i.e., every non-empty triangle of $\mathcal{X}_m$ contains exactly one point. Using Lemma 1, the number of non-convex 4-holes of $\mathcal{X}_m$ is three times the number of non-empty triangles.

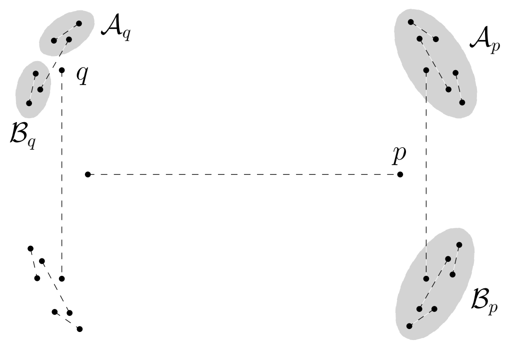

Figure 3: Example for $m = 4$ of the special point set defined in [10].

For each point of the set $\mathcal{X}_m$, we count the number of triangles that contain it. First, fix a point in the first layer $\mathcal{R}_1$, say p in Figure 3. Any triangle containing p is formed by one point of $\mathcal{A}_p$, one point of $\mathcal{B}_p$, and one point of the remaining set $\mathcal{C}_p = \mathcal{X}_m \backslash \{\mathcal{A}_p \cup \mathcal{B}_p \cup \{p\}\}$. We say that $\mathcal{A}_p$ and $\mathcal{B}_p$ are the *induced subsets* of p, and that $\mathcal{C}_p$ is the *remainder* (of $\mathcal{X}_m$) for p. As $a_1 := |\mathcal{A}_p| = |\mathcal{B}_p| = \frac{n-2}{4}$ and $c_1 := |\mathcal{C}_p| = n - 2 \cdot a_1 - 1 = \frac{n}{2}$, this gives $a_1^2 \cdot c_1$ triangles containing p, and thus the number of triangles containing a point of $\mathcal{R}_1$ is $2 \cdot a_1^2 \cdot c_1 = 2 \cdot (\frac{n-2}{4})^2 \cdot \frac{n}{2}$.

Now consider a point q in the second layer $\mathcal{R}_2$. Its induced subsets $\mathcal{A}_q$ and $\mathcal{B}_q$ have size $a_2 = \frac{n-6}{8}$, and the remainder $\mathcal{C}_q$ has $c_2 = n - 2 \cdot a_2 - 1 = \frac{3n+2}{4}$ points. In combination with $r_2 := |\mathcal{R}_2| = 4$ this gives a total of $4 \cdot (\frac{n-6}{8})^2 \cdot \frac{3n+2}{4}$ triangles containing a point of $\mathcal{R}_2$.

In general, $|\mathcal{R}_i| = r_i = 2^i$, and the size of the two induced subsets of any point p_i in $\mathcal{R}_i$ is

$$a_i = \frac{1}{r_{i+1}}(n - |\mathcal{X}_i|) = \frac{n - (2^{i+1}-2)}{2^{i+1}}.$$

Thus, with the size of the corresponding remainder $\mathcal{C}_{p_i}$ of

$$c_i = n - 2 \cdot a_i - 1 = \frac{(2^i - 1)n + 2^i - 2}{2^i},$$

we get $r_i \cdot a_i^2 \cdot c_i$ triangles containing one point of $\mathcal{R}_i$.

Using that every non-empty triangle of $\mathcal{X}_m$ gives three non-convex 4-holes, and summing up over all layers $\mathcal{R}_i$, we obtain

$$3 \cdot \sum_{i=1}^{m} r_i \cdot a_i^2 \cdot c_i =$$

$$3 \cdot \sum_{i=1}^{m} 2^i \left(\frac{n - (2^{i+1}-2)}{2^{i+1}}\right)^2 \frac{(2^i - 1)n + 2^i - 2}{2^i} =$$

$$\frac{1}{2}n^3 + \left(\frac{39}{4} - 3\log_2(n+2)\right) n^2 - O(n \log n)$$

for the total number of non-convex 4-holes of $\mathcal{X}_m$. ☐

5 Minimizing the Number of (General) 4-Holes

We obtained the following observation for general 4-holes by checking all corresponding point sets from the order type data base [2].

Observation 1 *Let S be a set of $n = 8$ points in the plane in general position, and $p_1, p_2 \in S$ two arbitrary points of S. Then S contains at least five 4-holes having p_1 and p_2 among their vertices.*

Based on this simple observation, we derive the following lower bound for the number of 4-holes.

Theorem 5 *Let S be a set of $n \geq 8$ points in the plane in general position. Then S contains at least $\frac{5}{2}n^2 - O(n)$ 4-holes.*

Proof. We consider the point set S in x-sorted order, $S = \{p_1, \ldots, p_n\}$, and sets $S_{i,j} = \{p_i, \ldots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ having at least 8 points is

$$\sum_{i=1}^{n-7} \sum_{j=i+7}^{n} 1 = \sum_{i=1}^{n-7} n - i - 6 = \frac{n^2}{2} - \frac{13}{2}n + 21.$$

By Observation 1, each set $S_{i,j}$ contains at least five 4-holes having p_i and p_j among their vertices (take the six points of $S_{i,j}$ which are nearest to the segment $p_i p_j$). Moreover, as p_i and p_j are the left- and rightmost points of $S_{i,j}$, they are also the left- and rightmost points for each of these 4-holes. This implies that any 4-hole of S counts for at most one set $S_{i,j}$, which gives a lower bound of $\frac{5}{2}n^2 - O(n)$ for the number of 4-holes in S. $\square$

Note that there exist sets which contain fewer than $1.94n^2$ convex 4-holes, while by the above result any set contains at least $2.5n^2$ (general) 4-holes.

6 Conclusion

We have shown lower and upper bounds on the numbers of (general) and non-convex 4-holes in point sets.

A natural generalization of this work is to consider similar questions for $k > 4$. For example, we have been able to show that for every constant $k \geq 3$, the maximum number of non-convex k-holes is asymptotically smaller than the maximum number of convex k-holes. This gives rise to the following conjecture, which is a general version of Theorem 2, and part of our ongoing research on this topic.

Conjecture 1 *For any constant $k \geq 5$ and sufficiently large n, the number of k-holes is maximized by a set of n points in convex position.*

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