Switch-regular Upward Planar Drawings with Low-degree Faces*

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Abstract

Upward planar drawings of digraphs are crossing free drawings where the vertices are mapped to points of the plane and all edges are simple curves that flow in the upward direction, according to their orientation. *Switch-regular upward planar drawings* are upward planar drawings with additional properties. They have both theoretical and practical interest in graph drawing. The complexity of deciding whether an embedded planar digraph admits a switch-regular upward planar drawing is still unknown. A linear time testing algorithm is known for digraphs whose underlying graph is a tree. In this paper we show that the switch-regular upward planarity testing problem can be solved in polynomial time also for planar embedded digraphs with faces of degree at most six.

1 Introduction

Let G be a planar digraph. An *upward planar drawing* of G is a planar drawing such that each vertex of G is mapped to a distinct point of the plane and all the edges of G are drawn as Jordan arcs monotonically increasing in the upward direction, according to their orientation. Not all planar digraphs admit an upward planar drawing, and the problem of deciding whether a planar digraph admits an upward planar drawing has been widely investigated in the literature. This problem is referred to as the *upward planarity testing problem*. In the variable embedding setting Garg and Tamassia showed that the upward planarity testing problem is NP-complete [12]; consequently, branch-and-bound algorithms and fixed-parameter tractable algorithms have been described [2, 7, 13]. Also, it has been shown that the problem can be solved in polynomial time for outerplanar digraphs [15], series-parallel digraphs [11], and single-source digraphs [4, 14]. In the fixed embedding setting, Bertolazzi *et al.* showed that the upward planarity testing can be executed in polynomial time by reducing it to a network flow problem [3].

The topological and combinatorial properties of upward planar drawings have also been investigated. In particular, Di Battista and Liotta discovered and characterized an interesting sub-family of upward planar drawings whose embedding has some interesting properties of "regularity" [9]. Roughly speaking, the *upward embedding* of an upward planar drawing describes the circular sequence of "small" angles (i.e., angles smaller than π) and "large" angles (i.e., angles larger than π) at the *switch vertices* of each face; a vertex v of a face f is a switch-vertex if its incident edges in f are both leaving or both entering v. The upward embedding of an upward planar drawing is said to be *switch-regular* if: (i) the boundary of each internal face contains at most one maximal subsequence of small angles of length greater than one; (ii) the external boundary does not contain two consecutive small angles. Figure 1 shows two different upward planar drawings of the same embedded planar digraph: The first drawing has a switch-regular upward embedding, while the second drawing has a non-switch regular upward embedding. An upward planar drawing with a switch-regular upward embedding is also called a *switch-regular upward planar drawing*.

Finding switch-regular upward planar drawings has practical relevance for the design of efficient graph drawing checkers [9] and of effective compaction algorithms [10]. The complexity of deciding whether a planar digraph admits a switch-regular upward planar drawing is still unknown, even in the fixed embedding setting. Binucci *et al.* proved that this problem can be solved in linear time in the variable embedding setting for digraphs whose underlying graph is a tree [5, 6].

In this paper we study the switch-regular upward planarity testing problem in the fixed embedding setting. We prove that, given an embedded planar digraph G with n vertices and face-degree at most six, it is possible to decide in $O(n^2)$ time whether G admits a switch-regular upward planar drawing, and to compute one in the positive case.

The remainder of the paper is structured as follows. In Section 2 we recall some basic definitions and results about upward planarity and switch-regular embeddings. In Section 3 we present our testing and embedding algorithm. Some open problems are listed in Section 4.

2 Basic definitions and results

We assume familiarity with the basic concepts of graph drawing and graph planarity [8]. Let G be an embedded planar digraph. A vertex v of G is *bimodal*

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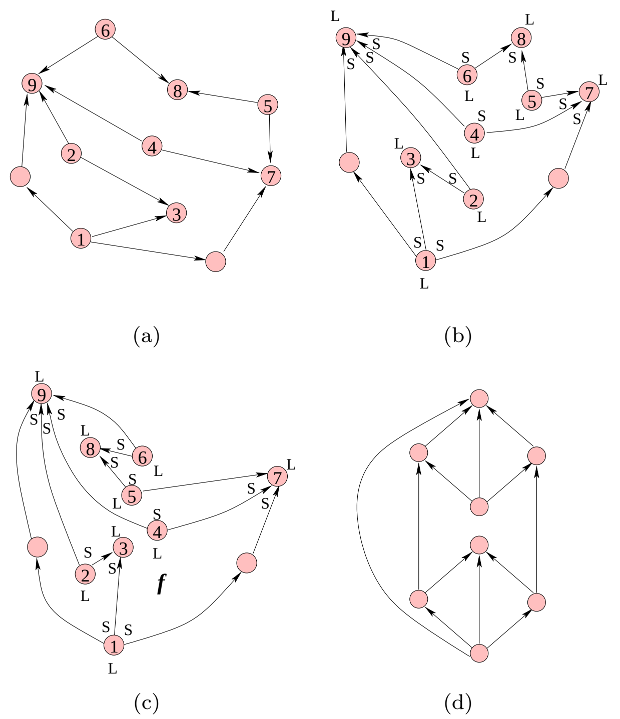

Figure 1: (a) A planar embedded digraph G; source and sink vertices are labeled with integer numbers. (b) An upward planar drawing of G whose upward embedding is switch-regular; S and L labels denote small and large angles, respectively; (c) An upward planar drawing of G whose upward embedding is not switch-regular (the external face and the internal face f are not switch-regular). (d) A planar digraph that does not admit a switch-regular upward planar drawing (the internal face of degree six is not switch-regular in any upward planar drawing of the digraph).

if all incoming edges of v (and hence all outgoing edges of v) are consecutive in the circular clockwise order around v. G is called *bimodal* if all its vertices are bimodal. Acyclicity and bimodality are necessary, but not sufficient, conditions for the upward planar drawability of an embedded planar digraph [3].

Let f be a face of G and suppose that the boundary of f is traversed counterclockwise. If $e_1 = (u_1, v)$ and $e_2 = (v, u_2)$ are two edges encountered in this order along the boundary of f, the triplet $s = (e_1, v, e_2)$ is called an *angle of* f. Observe that, e_1 and e_2 may coincide if G is not biconnected. Angle s is called a *switch of* f if e_1 and e_2 are both incoming edges or both outgoing edges of v: In the first case s is also called a *sink-switch of* f, while in the second case it is a *source-switch of* f. It is immediate to see that the number of source-switches of f equals the number of sink-switches of f. We denote by $2n_f$ the total number of switches of f. The *capacity of* f is denoted by c_f and it is defined by $c_f = n_f - 1$ if f is an internal face and by $c_f = n_f + 1$ if f is the external face. For a bimodal digraph G, an assignment of its sources and sinks to the faces of G is called an *upward consistent assignment* if: (i) Each source and each sink is assigned to exactly one of its incident faces; (ii) for each face f exactly c_f sources and sinks on the boundary of f are assigned to f. The following theorem characterizes the class of embedded planar digraphs that admit an upward planar drawing.

Theorem 1 [3] *An acyclic embedded planar bimodal digraph is upward planar if and only if it admits an upward consistent assignment.*

The *upward embedding* corresponding to an upward consistent assignment of G is a planar embedding of G with labels at the switches of every face. Namely, a switch $s = (e_1, v, e_2)$ of f is labeled L when v is a source or a sink assigned to f, while s is labeled S otherwise. If f is a face of an upward embedding, the circular list of labels of f is denoted by σ_f. Also, S_{σ_f} and L_{σ_f} denote the number of S and L labels of f, respectively. The following property holds.

Property 1 [3] *If f is a face of an upward embedding then $S_{\sigma_f} = L_{\sigma_f} + 2$ if f is internal, and $S_{\sigma_f} = L_{\sigma_f} - 2$ if f is external.*

An upward planar drawing of a digraph G can be constructed for a given upward embedding of G; this drawing is such that each switch labeled L forms a geometric angle larger than π, while each switch labeled S forms a geometric angle smaller than π.

An internal face f of an upward embedding is called *switch-regular* if σ_f does not contain two distinct maximal subsequences σ_1 and σ_2 of S labels such that $S_{\sigma_1} > 1$ and $S_{\sigma_2} > 1$. The external face f is *switch-regular* if σ_f does not contain two consecutive S labels. An upward embedding is *switch-regular* if all its faces are switch-regular. We say that a digraph G is *switch-regular* if it admits a switch-regular upward embedding.

Figure 1 shows examples of switch-regular and non-switch regular upward embeddings for the same digraph G. Both the upward embeddings preserves the planar embedding of G. Figure 1(d) shows a non-switch regular digraph.

3 Digraphs with face-degree at most six

Here, we present a polynomial-time testing algorithm for embedded planar digraphs whose faces have at most six switches. Clearly, this condition is always true if the faces have degree at most six.

We reduce the switch-regular upward planarity testing problem on G to the computation of an integer feasible flow in a suitable network $N(G)$ constructed from G. Our flow model is inspired by that proposed by Bertolazzi *et al.* for the classical upward planarity testing problem [3]. However, our network is enhanced with extra elements (nodes and arcs), in order to guarantee that every face is assigned a switch-regular embedding when possible.

The flow network $N'(G)$ of Bertolazzi *et al.* is based on Theorem 1. It consists of a node $n(u)$ for each source or sink vertex u of G and of a node $n(f)$ for each face f of G. Vertices $n(u)$ and $n(f)$ are called *vertex-nodes* and *face-nodes*, respectively. Also $N'(G)$ has a directed arc $a(u,f) = (n(u), n(f))$ if u belongs to f; the arc has upper capacity one and lower capacity zero. A unit of flow on an arc $a(u,f)$ represents a large angle formed at u inside f (i.e., a label L at u inside f). Each vertex-node $n(u)$ supplies flow one, which represents the fact that u must form exactly one large angle inside one of its incident faces. Each face-node $n(f)$ demands an amount of flow equal to the capacity c_f of face f, which represents the fact that f must arrange exactly c_f large angles. Hence, there exists an upward consistent assignment describing an upward embedding of G if and only if $N(G')$ admits an integer feasible flow.

We construct a flow network $N(G)$ by enhancing $N'(G)$. First we want to restrict all possible upward consistent assignments to those assignments whose corresponding upward embeddings have switch-regular internal faces only. If f is an internal face of G, we distinguish between two possible cases:

f has either 2 or 4 switch-angles. In this case c_f is either 0 or 1, and f has to arrange at most one large angle. Hence, f is switch-regular in every upward embedding of G. No local modification of the network is required for this case.

f has 6 switch-angles. In this case $c_f = 2$ and then f has to arrange two large angles. There are three forbidden configurations, shown in Figure 2, that we have to avoid. To this aim, we locally modify the network as follows (see also Figure 2(d)): If u and v are sources or sinks incident to f, such that there are two switch-angles along the boundary of f between u and v (in both the two circular directions), then we remove arcs $a(u,f)$ and $a(v,f)$ and add the following extra elements: (i) A node $n_{uv}(f)$ that neither supplies nor demands flow (it is a transient node); (ii) Two directed arcs $(u, n_{uv}(f))$ and $(v, n_{uv}(f))$, each having upper capacity one and lower capacity zero. (iii) A directed arc $(n_{uv}(f), n(f))$, with upper capacity one and lower capacity zero.

Since $n_{uv}(f)$ is a transient node and the upper capacity of arc $(n_{uv}(f), n(f))$ is one, such a local modification of the network avoids that both u and v receive an L label inside f, but leaves the possibility that one of them has such a label.

For the external face h of G, we must avoid those

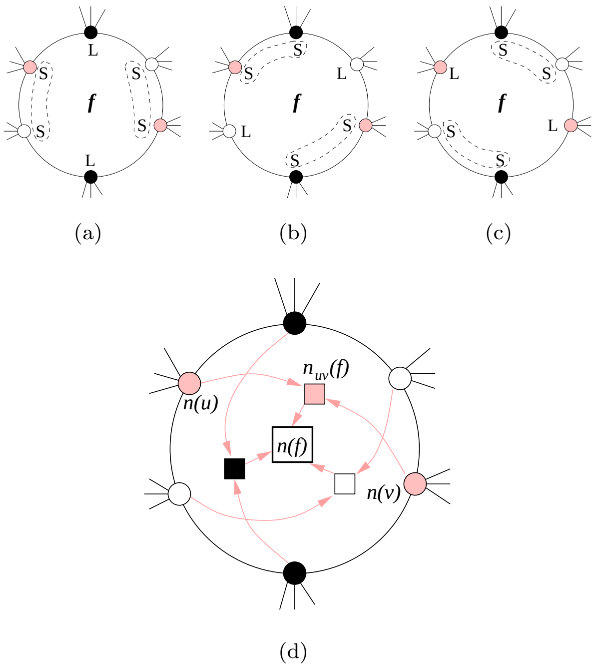

Figure 2: (a)–(c) The three forbidden upward embeddings for an internal face with six switches. Maximal subsequences of S labels are bounded by dashed closed curves. (d) Flow network for an internal face with six switches. All arcs have lower capacity zero and upper capacity one.

upward embeddings having two consecutive S labels along the boundary of h. If h has less than six switches, then h is switch-regular in any upward embedding of G. If h has six switches, then $c_h = 4$ and h has to arrange four L labels and two S labels. Hence, there are at most 15 possible arrangements of S and L labels for the switches of h, 9 of which correspond to switch-regular upward embeddings of h. For each of these switch-regular upward embeddings, we can locally fix in $N(G)$ the flow associated with that upward embedding, and look for an integer feasible flow within this constraint. To fix a flow of value x on an arc of $N(G)$, it is sufficient to set upper and lower capacity of that arc equal to x.

Figure 3 shows the flow network for the embedded digraph of Figure 1(a) and a feasible flow corresponding to the switch-regular upward embedding of Figure 1(b).

Theorem 2 *Let G be an embedded planar digraph with n vertices, such that every face has at most six switches. There exists an $O(n^2)$ time algorithm that tests whether G admits a switch-regular upward planar drawing and that computes such a drawing in the positive case.*

Proof. Construct the flow network $N(G)$ from the planar embedding of G. This is done in $O(n)$ time.

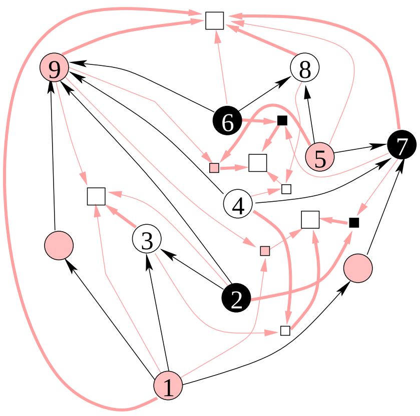

Figure 3: The flow network for the embedded digraph of Figure 1(a) (grey arcs). The flow in the network corresponds to the switch-regular embedding of Figure 1(b). Thick arcs have flow one, while the others have flow zero.

For each possible switch-regular upward embedding of the external face h of G, fix the corresponding flow on $N(G)$ and look for an integer feasible flow in $N(G)$. If at least one feasible flow is found, then the test is positive, otherwise it is negative. Since both the size of $N(G)$ and the value of the flow are $O(n)$, an integer feasible flow in $N(G)$ can be computed in $O(n^2)$ time with a standard algorithm [1]. Also, we need to run such an algorithm a constant number of times, because there are at most 9 switch-regular upward embeddings for h. Hence, it is possible to test the existence of an integer feasible flow corresponding to a switch-regular upward embedding of G in $O(n^2)$ time. If such a flow exists, the label assignment associated with its values describes a switch-regular upward embedding of G, and an upward planar drawing with this upward embedding is computed with the same linear time procedure described in [3]. □

Corollary 1 *Let G be an embedded planar digraph with n vertices and face-degree at most six. There exists an $O(n^2)$ time algorithm that tests whether G admits a switch-regular upward planar drawing and that computes such a drawing in the positive case.*

4 Open Problems

The main open question is what is the complexity of the switch-regular upward planarity testing problem in the general case. It is also interesting to extend the approach presented in this paper to design a fixed parameter tractable algorithm whose time complexity depends on the maximum face degree.

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