The Traveling Salesman Problem with Differential Neighborhoods

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Abstract

We introduce a novel differential approach to the Traveling Salesman Problem with Disk Neighborhoods (TSPDN), in which each node may be relocated within radius r from the original location in order to decrease the length of the shortest tour visiting all nodes. When r is small compared to the distance between the nodes, the optimal solution to the TSPDN is achieved by shortening the cycle corresponding to the optimal TSP tour without reordering the nodes. Looking at the shortening rate of a cycle, defined as the ratio of the decrease in tour length to r when r tends to zero, gives us an insight on how the movement of nodes can be converted into savings in tour length. We study the optimal direction for shortening and show how the shortening rate relates to the tightness of turns, number of U-turns and the distance from the origin in Euclidean, Manhattan and hyperbolic metrics, respectively.

1 Introduction

A natural generalization of the Traveling Salesman Problem (TSP) is the TSP with neighborhoods (TSPN): given a collection of n regions (disks, rectangles, lines, ...), the goal is to find a shortest tour that visits all of them [2, 5].

In this work, neighborhoods are defined as disks centered at the initial locations of nodes to be visited by the salesman and the radius r of the disks denotes the maximum distance each node can be shifted in order to decrease the total tour length. We may equivalently think that there are n customers and r denotes the maximum distance each customer can walk to meet the salesman. We are interested in studying the effect of relocating nodes on total tour length, by comparing the optimal TSP tour to the solution of the TSP with disk neighborhoods (TSPDN).

While the absolute difference between the solutions to TSP and TSPDN is clearly increased with r, the most efficient "steps" are taken when r is small. This is seen to be true also for arbitrary cycles, consisting of a finite number of nodes, that are shortened by shifting each node a distance r in the optimal direction. We examine the *shortening rate* of a cycle, defined as the ratio of the decrease in the length of the cycle to r when $r \to 0$. This approach is in fact similar to discrete curve shortening flow models presented in [3, 12, 16, 18]. Most of these works focus on the asymptotic behavior of polygons that evolve according to a specific shortening flow (see section 1.1). In contrast to these models, we assume that all nodes move at equal "speed", that is, r is equal for each node, and focus on the optimal direction for shortening and the corresponding maximum shortening rate. By shifting the nodes in an optimal TSP tour a differential distance dr in the optimal direction, we achieve an optimal solution to the TSP with differential neighborhoods.

The main contributions are Theorems 3, 4 and 5, which characterize the optimal direction for shortening and the shortening rate in arbitrary cycles in different metrics: The shortening rate is dependent on the sharpness of turns in the Euclidean metric, number of U-turns in the Manhattan metric and the distance from the origin in the hyperbolic metric. Furthermore, we derive global bounds for the shortening rate in arbitrary cycles and upper bounds for the shortening rate in optimal TSP tours.

This work is partially motivated by different types of vehicle routing problems, where the shortening rate translates to the available savings in mileage given the nodes (goods, passengers, customers of a mobile service provider) are shifted in optimal directions.

Proofs are omitted due to space limitations. However, we give a short description of each proof.

1.1 Related work

In the Group-TSP [7], also known as the One-of-a-Set-TSP [14] and the Errand Scheduling problem [17], a salesman has to meet n customers, to each of which is associated a set of at most k possible meeting places. If the weights are symmetric, the optimal solution can be polynomially approximated with ratio $3k/2$ [17].

The Group-TSP in which the neighborhoods are connected regions in the plane is referred to as TSP with neighborhoods (TSPN). In [11], a polynomial time algorithm with approximation ratio $O(n^2 \log n)$ for the general TSPN is provided, where n is the number of neighborhoods. In [2], constant factor approximation algorithms are presented for special cases in which the neighborhoods are defined as parallel unit segments, translations of convex polygonal neighborhoods and circles. These results are extended in [6] to neighborhoods with comparable diameter, unit disks, and infinite lines. In [11], a polynomial time approximation scheme (PTAS) is introduced for the special case where the tour is short compared to the size of the neighborhoods. In [15], a PTAS for the TSPN with disjoint fat regions in the plane is presented.

In addition to the TSPN, our approach is closely related to curve shortening flow [8, 9, 4], in the Euclidean version

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of which the length of a closed continuous curve is decreased by shifting each point on the curve along the normal vector at a rate proportional to curvature. This method shrinks the length of a curve as fast as possible using only local information [10].

The results obtained in the curve shortening literature have motivated research in creating discrete analogues of the flows [18]. Reference [3] introduces a linear polygon shortening flow where each vertex chases the centroid of its two neighboring vertices, which shrinks polygons to elliptical points. In addition, [3] proposes a Euclidean polygon shortening scheme based on the Menger-Melnikov curvature [13]. In [12] it is shown that in this case most quadrilaterals shrink to circular points. In [16], a discrete curve shortening equation is formulated such that the perimeter of the polygon is monotonically decreasing.

For the linear polygon shortening flow, proposed in [3], it is shown in [18] that 1) polygons shrink to elliptical points, 2) convex polygons remain convex, 3) if vertices are arranged in a star formation about their centroid, they will remain in a star formation for all time and 4) the perimeter of the polygon is a monotonically decreasing function of time. In addition, the authors derive the optimal direction for perimeter shortening.

2 Preliminaries

In the following, $S = (s_1, \dots, s_n)$ denotes a cycle of n distinct nodes in a metric space (X, d) and $|S| = \sum_{i=1}^{n} d(s_{i-1}, s_i) \pmod{n}$ denotes the length of the cycle with respect to metric d. $S(P)$ denotes the cycle corresponding to the optimal solution to the classical traveling salesman problem with the node set $P = \{p_1, \dots, p_n\}$. Moreover, $L_S(r)$ denotes the minimum length of all cycles $(q_1, \dots, q_n)$ for which $d(q_i, s_i) \le r$ for all i.

If the nodes of the problem are relatively far apart from each other compared to r, the TSPDN may be solved up to optimality by improving the optimal center tour (the solution to the TSP) without reordering nodes. In this case, we say that the set of nodes is r-stable.

It is easy to see that any fixed node set for which the minimum distance between two nodes is $\epsilon > 0$, will become r-stable when r is decreased (see Figure 1).

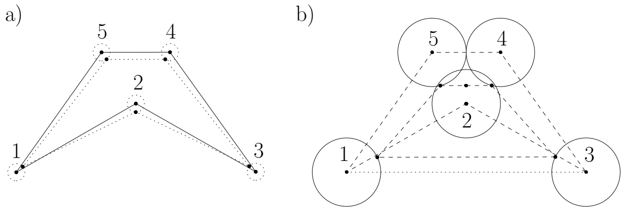

Figure 1: Stability of TSPDN. The solid lines represent the optimal solution to the classical TSP (center tour) and the dashed lines represent the optimal solution to the r-TSPDN. As r is decreased, the two solutions converge.

In this work, we focus on the difference between TSP and TSPDN when r tends to zero. Thus, the optimal ordering of the TSP is preserved.

3 Analysis

The shortening rate in a cycle S is defined by $-L'_S(0) = \lim_{r\to 0} \frac{L_S(0)-L_S(r)}{r}$, which is equal to the decrease in the length of the cycle, relative to the differential radius of the neighborhoods. $L'_S(0)$ corresponds to the change in the length of S when each customer takes a differential step dr in the optimal direction. Note that $-L'_S(0) \ge 0$, since $L_S(r) \le L_S(0)$ due to the fact that the nodes of the center tour $L_S(0)$ are always included in the neighborhoods of radius $r \ge 0$. A large absolute value of $L'_S(0)$ means that the shortening rate is significant. In the following, we will study the shortening rate in arbitrary cycles with Euclidean, Manhattan and hyperbolic metrics.

3.1 Global bounds

Theorem 1 *Let $S = (s_1, \dots, s_n)$ be a cycle in a metric space (X, d). Then, $L_S(0) - L_S(r) \le 2nr$ for all $r \ge 0$.*

Theorem 1 states that no cycle can be shortened by more than than twice the total walking distance of customers. This is proved by adding a detour of at most $2r$ from each optimal meeting location to the original location. For the case studied in [6], in which the neighborhoods are defined as unit disks, the theorem can be applied as follows: For disjoint unit disks, Theorem 1 implies $L_S(0) - L_S(1) \le 2n \le L_S(0)$. By looking at the corresponding inequality derived from [6], namely $L_S(0) - L_S(1) \le |L_S(0)| \frac{8+8\pi/L_S(0)}{\pi+8}$, it can be seen that our result improves this bound whenever the length $L_S(0)$ of the center tour satisfies $L_S(0) < 8$. For arbitrary unit disks, we derive from [6] the inequality $L_S(0) - L_S(1) \le |L_S(0)| \frac{\pi+7+10\pi/L_S(0)}{\pi+8}$. Our upper bound $2n$ improves this result whenever $2n < L_S(0) < 10\pi$.

Consider next the corresponding lower bound for $L_S(0) - L_S(r)$. For all cycles visiting $n \ge 2$ disks centered on a straight line, of which at least two are disjoint, we have $L_S(0) - L_S(r) = 4r$. For the general case, we obtain the following result by considering a minimum bounding circle of radius R and scaling it down by $(1 - \frac{r}{R})$.

Theorem 2 *Let $S = (s_1, \dots, s_n)$ be a cycle in $\mathbb{R}^2$ satisfying $\max_{i,j\in\{1,\dots,n\}} \|s_i - s_j\| \ge 2r \ge 0$. The minimum length $L_S(r)$ satisfies $L_S(0) - L_S(r) \ge 4r$ in the Euclidean (L^2) and Manhattan (L^1) metrics.*

3.2 Euclidean metric

In [18], it has been shown that for a planar Euclidean cycle, the optimal *direction* for differential perimeter shortening is towards the bisector of the angle between two subsequent legs of the cycle. In the following, we examine the corresponding *shortening rate* assuming that all nodes move at equal speed in the optimal direction.

Theorem 3 *Let $S = (s_1, \dots, s_n)$ be a cycle in $\mathbb{R}^2$ with angles α_i at corner s_i. In the Euclidean metric, the shortening rate of S is given by the formula $-L'_S(0) = 2\sum_{i=1}^{n} \cos\frac{\alpha_i}{2}$.*

This result, which is proved by considering a sequence of intermediate cycles, states that the shortening rate depends only on the angles between subsequent legs of a cycle. Assuming that the angles in a cycle S are uniformly distributed between 0 and π, the expected shortening rate in the cycle is given by $E(-L'_S(0)) = 2n\int_0^{\pi}\frac{1}{\pi}\cos\frac{\alpha}{2}d\alpha \approx 1.273n$. Furthermore, since $2\cos 60° = 1$, it can be stated that if the angle corresponding to a node i in a cycle is less than 120°, then the improvement in the length of the cycle achieved by relocating i is asymptotically greater than the walking distance r.

3.3 Manhattan metric (L^1-norm)

Definition 1 *A sequence of nodes $(s_h, s_{h+1}, \dots, s_k)$ in a cycle $S = (s_1, \dots, s_n) = ((x_1, y_1), \dots, (x_n, y_n)) \pmod n$ in $\mathbb{R}^2$ is a U-turn with respect to x if $x_h = x_{h+1} = \dots = x_k$ and $\operatorname{sgn}(x_h - x_{h-1}) = \operatorname{sgn}(x_k - x_{k+1}) \neq 0$.*

A U-turn with respect to y is defined similarly. Two U-turns X and Y are called disjoint, if $X \cap Y = \emptyset$. The following theorem establishes a relation between the number of U-turns and the shortening rate.

Theorem 4 *Let $S = (s_1, \dots, s_n)$ be a cycle in $\mathbb{R}^2$ and let M denote the minimum depth of U-turns in S. If $r < M/2$, the length of the cycle in the Manhattan metric satisfies $L_S(0) - L_S(r) = 2U(S)r$, where $U(S)$ is the maximum number of disjoint U-turns in S.*

This result is proved similarly as Theorem 3. For disjoint disk neighborhoods of radius r, the contribution of each U-turn on the decrease of the length of a cycle is $2r$. If all nodes are U-turns, a cycle can be shortened by $2nr$, which is an upper bound for all cycles. Differentiating the result yields the shortening rate $-L'_S(0) = 2U(S)$. For example, the expected shortening rate for a random cycle S in $\mathbb{R}^2$ equals $E(-L'_S(0)) = 2\cdot\frac{3}{4} = \frac{3}{2}n$.

Clearly, the shortening rate is invariant to linear transformations since the number of U-turns depends only on the signs of the differences between subsequent nodes. More precisely, applying a linear function $f(x, y) = (a_x + b_x x, a_y + b_y y)$, where $b_x, b_y \neq 0$, on the points of any cycle will not affect the number of U-turns.

Example 1 *The expected number of U-turns in a random cycle S in $[0, a]\times[0, b] \in \mathbb{R}^2$, consisting of n nodes equals $E(U(S)) = \frac{n}{ab}\int_0^b\int_0^a P(U \mid x, y)\,dxdy = \frac{8}{9}n$. The expected shortening rate thus equals $E(-L'_S(0)) = \frac{16}{9}n \approx 1.78n$, independent of the ratio of a to b.*

For a random cycle S in an ellipse with half axes a and b, we get $E(-L'_S(0)) = \frac{n}{6}\left(11 - \frac{2}{\pi^2}\right) \approx 1.80n$. Again, $E(-L'_S(0))$ is independent of the eccentricity of the ellipse.

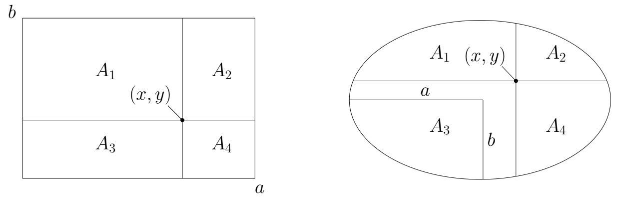

Figure 2: Illustration of Example 1. The probability that a node located in (x, y) is a U-turn in a random cycle is given by $P(U \mid x, y) = 1 - \frac{2}{(A_1+A_2+A_3+A_4)^2}(A_1A_4 + A_2A_3)$.

3.4 Hyperbolic metric

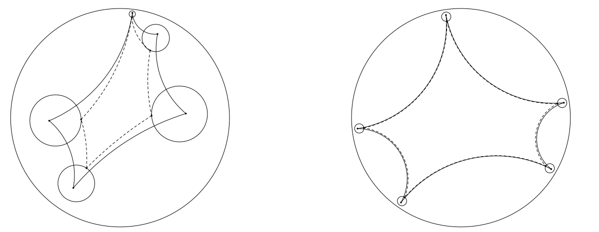

Figure 3: The TSPDN in the Poincaré disk model. The solid and dashed lines represent the optimal center tour and a solution to the TSPDN, respectively.

In hyperbolic geometry, all postulates of Euclidean geometry are satisfied except the parallel postulate. In the Poincaré hyperbolic disk [1], the hyperbolic distance between two points u, v in the unit disk is defined by the formula $d(u, v) = \operatorname{arccosh}\left(1 + \frac{2\|u-v\|^2}{(1-\|u\|^2)(1-\|v\|^2)}\right)$, where $\|\cdot\|$ denotes the Euclidean norm. If $\|u\|$ and $\|v\|$ are small, $d(u, v)$ can be approximated by $2\,\|u - v\|$. The hyperbolic disk neighborhood with hyperbolic radius $\rho \geq 0$ of a node u consists of points v for which $d(u, v) \leq \rho$. Note that the hyperbolic disk corresponds to a disk in the Euclidean plane, but the Euclidean center of the disk is different from the hyperbolic center u unless $u = (0, 0)$ (see reference [1]). Let us consider a solution to the hyperbolic TSPDN obtained by shifting each node in the optimal center tour an equal hyperbolic distance ρ towards the origin (see Figure 3), which gives us a lower bound for the shortening rate.

Theorem 5 *Let $S = (s_1, \dots, s_n)$ be a cycle in the unit disk and $r_i = \|s_i\|$ for $i \in \{1, \dots, n\}$, where $\|\cdot\|$ denotes the Euclidean metric. The shortening rate of S in the hyperbolic metric satisfies $-L'_S(0) \geq \sum_{i=1}^{n}(r_i + r_{i+1})\left(1 - \frac{|r_i - r_{i+1}|}{\|s_i - s_{i+1}\|}\right)$, where $s_{n+1} = s_1$ and $r_{n+1} = r_1$.*

This result is proved by differentiating the hyperbolic length of leg (s_i, s_{i+1}) with respect to r. If all nodes are located at an equal Euclidean distance x from the origin, we have $-L'_S(0) \geq \sum_{i=1}^{n} 2x = 2xn$. Since by Theorem 1, the shortening rate always satisfies $-L'_S(0) \leq 2n$, our approximate solution is asymptotically optimal when the nodes approach the border of the unit disk.

Corollary 6 *Let $S_2, S_3, \ldots$ be a sequence of cycles of n nodes, where $S_j = ((1 - 1/j, \theta_1), \ldots, (1 - 1/j, \theta_n))$ in polar coordinates for all $j \geq 2$. Then, $\lim_{j\to\infty} -L'_{S_j}(0) = 2n$.*

3.5 Bounds for TSP tours

The shortening rate is generally governed by the tightness and number of turns in a cycle. In some cases, even the optimal TSP tour includes relatively tight turns. We state the following for Euclidean TSP tours in the plane.

Conjecture 1 *Let P be a node set in $\mathbb{R}^2$ involving $n > 2$ customers. The length of the shortest tour $S(P)$ satisfies $L_{S(P)}(0) - L_{S(P)}(r) \leq \sqrt{3}nr$ in the Euclidean metric.*

While the inequality in Conjecture 1 has not been proven to be valid, there exist no known problems P for which $-L'_{S(P)}(0) > \sqrt{3}n$. The equality is achieved for $n = 3$ with an equilateral triangle. In addition, it is possible to construct a sequence of problems that asymptotically satisfies the equality. For Manhattan TSP tours in the plane, the upper bound is equal to that of arbitrary cycles.

Theorem 7 *There exists (i) a sequence of problems $P_4, P_6, P_8, \ldots$ such that $\lim_{k\to\infty} \frac{1}{2k}(-L'_{S(P_{2k})}(0)) = \sqrt{3}$ in the Euclidean metric and (ii) for any $r \geq 0$ a sequence of problems $P_4, P_{12}, P_{20}, \ldots$ such that $L_{S(P_n)}(0) - L_{S(P_n)}(r) = 2nr$ for all $n = 4(2k - 1), k \in \mathbb{N}$ in the Manhattan metric, where $|P_j| = j$ for all $j \in \mathbb{N}$.*

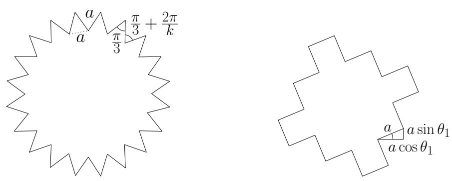

Figure 4: Theorem 7. The left figure shows a structure in which the shortening rate per node in the optimal Euclidean solution approaches $\sqrt{3}$ as $n \to \infty$. The right figure shows an optimal Manhattan TSP tour where each node is a U-turn and the cycle can be shortened by $2nr$.

Theorem 7 suggests that optimal Manhattan TSP tours may be shortened more efficiently than Euclidean TSP tours. In the hyperbolic metric, we know by Corollary 6 that for any sequence of cycles where the nodes approach the border of the unit disk, the shortening rate converges to $2n$. In this case, our solution approach to the hyperbolic TSPDN converges to the optimal solution.

4 Conclusions

In this work we study the difference between the length of an optimal TSP tour and the length of the optimal solution to the TSP with disk neighborhoods, in which each node can be redirected to a new location within a certain radius r from the original location. We show that the shortening rate is characterized by the tightness of turns in the Euclidean metric and it is equal to two times the number of U-turns with respect to the coordinate axes in the Manhattan metric. In the hyperbolic metric, the shortening rate increases with the distance of nodes from the origin.

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