Two-Site Voronoi Diagrams under Geometric Distance Functions

Gill Barequet* Matthew T. Dickerson† David Eppstein‡ David Hodorkovsky§ Kira Vyatkina¶

Abstract

We revisit a new type of a Voronoi diagram, in which distance is measured from a point to a *pair* of points. We consider a few more such distance functions, and analyze the structure and complexity of the nearest- and furthest-neighbor Voronoi diagrams of a point set with respect to these distance functions.

1 Introduction

The Voronoi diagram is a beautiful geometric structure, which has a wide variety of applications in the most diverse areas. Detailed surveys of its history, applications, and variants are given by Aurenhammer [4] and by Okabe, Boots, and Sugihara [11]. One of the recent generalizations of this concept is a family of so-called *2-site Voronoi diagrams* [5], which are based on distance functions that define a distance from a point in the plane to a *pair* of sites from a given set S. Consequently, each Voronoi region corresponds to an (unordered) pair of sites from S.

For S being a set of points, Voronoi diagrams under a number of 2-site distance functions have been investigated, which include arithmetic combinations of point-to-point distances [5, 13] and certain geometric distance functions [5, 7, 8]. In this work, we develop further the latter direction.

Let $S \subset \mathbb{R}^2$, and consider $p, q \in S$ and a point v in the plane. We shall focus our attention on a few circle-based distance functions:

- *radius of circumscribing circle*:
  $\mathcal{C}(v,(p,q)) = \mathrm{Rad}(\circ(v,p,q))$, where $\circ(v,p,q)$ is the circle defined by v, p, q,
- *radius of containing circle*:
  $\mathcal{K}(v,(p,q)) = \mathrm{Rad}(C(v,p,q))$, where $C(v,p,q)$ is the minimum circle containing v, p, q,
- *view angle*:
  $\mathcal{V}(v,(p,q)) = \measuredangle pvq$, or, equivalently, half of the angular measure of the arc of $\circ(v,p,q)$ that the angle $\measuredangle pvq$ subtends,

and on a parameterized perimeter distance function:

- *parameterized perimeter*:
  $\mathcal{P}_c(v,(p,q)) = |vp| + |vq| + c \cdot |pq|$, where $c \geq -1$.

The first and the third circle-based distance functions were first mentioned in [9]. The last function generalizes the perimeter distance function $\mathcal{P}(v,(p,q)) = \mathrm{Per}(\triangle(v,p,q)$ introduced in [5], and later addressed in [7, 8].

Since two points define a segment, any 2-point site distance function $d(v,(p,q))$ provides a distance between the point v and the segment pq, and vice versa. Consequently, geometric structures akin to 2-site Voronoi diagrams can arise as Voronoi diagrams of segments. This alternative approach was independently undertaken by Asano et al., and the "view angle" and the "radius of circumscribing circle" distance functions reappeared in their works [2, 3] on Voronoi diagrams for segments soon after they had been proposed by Hodorkovsky [9] in the context of 2-site Voronoi diagrams. However, as Asano's et al. research was originally motivated by mesh generation and improvement tasks, they were mostly interested in sets of segments representing edges of a simple polygon, and thus, non-intersecting (except, possibly, at the endpoints), what significantly alters the essence of the problem.

In this paper, we analyze the structure and complexity of 2-site Voronoi diagrams under the distance functions listed above. Our obtained results are mostly of theoretical interest. The method used to derive an upper bound on the complexity of the 2-site Voronoi diagram under the "parameterized perimeter" distance function is first developed for the case of $c = 1$, yielding a much simpler proof for the "perimeter" function than the one developed in [8], and then generalized to any $c \geq 0$.

Throughout the paper we use the notation $V_{\mathcal{F}}^{(n)}(S)$ (resp., $V_{\mathcal{F}}^{(f)}(S)$) for denoting the nearest- (resp., furthest-) 2-site Voronoi diagram, under the distance function $\mathcal{F}$, of a point set S. The set S is always assumed to contain n points.

2 Circumscribing Circle

Let $\circ(p,q,r)$ denote the unique circle defined by three distinct points p, q, and r in the plane.

*Dept. of Computer Science, The Technion—Israel Institute of Technology, Haifa 32000, Israel. E-mail: barequet@cs.technion.ac.il

†Dept. of Mathematics and Computer Science, Middlebury College, Middlebury, VT 05753. E-mail: dickerso@middlebury.edu

‡Dept. of Information and Computer Science, University of California, Irvine, CA 92717. E-mail: eppstein@ics.uci.edu

§Dept. of Applied Mathematics, The Technion—Israel Institute of Technology, Haifa 32000, Israel. Currently affiliated with Imagine Communications, Ltd. E-mail: david@imagine-com.com

¶Dept. of Mathematics and Mechanics, Saint Petersburg State University, 28 Universitetsky pr., Stary Peterhof, St. Petersburg 198504, Russia, E-mail: kira@math.spbu.ru, and Dept. of Natural Sciences, Saint Petersburg State University of Information Technologies, Mechanics and Optics, 49 Kronverkskiy pr., St. Petersburg 197101, Russia.

Definition 1 *Given two points p, q, the "circumcircle distance" $\mathcal{C}$ from a point v to the unordered pair (p, q) is defined as $\mathcal{C}(v, (p, q)) = \mathrm{Rad}(\circ(v, p, q))$.*

For a fixed pair of points p, q, the curve $\mathcal{C}(v, (p, q)) = \infty$ is the line $\overline{pq}$. This implies that all the points on $\overline{pq}$ belong to the region of (p, q) in $V^{(f)}_{\mathcal{C}}(S)$. In this section we assume that the points in S are in general position, i.e., there are no three collinear points, and no three pairs of points define three distinct lines that intersect at one point. The given sites are singular points, i.e., for any two sites p, q, the function $\mathcal{C}(v, (p, q))$ is not defined at $v = p$ or $v = q$.

Theorem 1 *Let S be a set of n points. The complexity of $V^{(f)}_{\mathcal{C}}(S)$ is $\Omega(n^4)$.*

Proof. The n points define $\Theta(n^2)$ lines with $\Theta(n^4)$ intersection points. All these intersection points are features of $V^{(f)}_{\mathcal{C}}(S)$, and hence the lower bound. □

Theorem 2 *Let S be a set of n points. The complexity of both $V^{(n)}_{\mathcal{C}}(S)$ and $V^{(f)}_{\mathcal{C}}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).*

Proof. The complexity of $V^{(n|f)}_{\mathcal{C}}(S)$ is identical to that of the respective diagram of $\mathcal{C}^2(v, (p, q)) = \mathrm{Rad}^2(\circ(v, p, q))$. It is known that $\mathrm{Rad}^2(\circ(v, p, q)) = ((|vp||vq||pq|)/(4|\triangle vpq|))^2 = (((v_x - p_x)^2 + (v_y - p_y)^2)((v_x - q_x)^2 + (v_y - q_y)^2)((p_x - q_x)^2 + (p_y - q_y)^2))/(4(v_x(p_y - q_y) - p_x(v_y - q_y) + q_x(v_y - p_y))^2)$. The respective collection of $\Theta(n^2)$ Voronoi surfaces fulfills Assumptions 7.1 of [12, p. 188]: (1) Each surface is an algebraic surface of maximum constant degree; (2) Each surface is totally defined (stronger than needed); and (3) Each triple of surfaces intersects in $O(1)$ points. Hence, we may apply Theorem 7.7 of [ibid., p. 191] and obtain the claimed bound. □

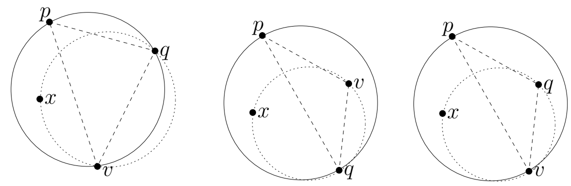
3 Containing Circle

Let $C(p, q, r)$ denote the minimum-radius circle containing three points p, q, r.

Definition 2 *Given two points p, q, the "containing-circle distance" $\mathcal{K}$ from a point v to the unordered pair (p, q) is defined as $\mathcal{K}(v, (p, q)) = \mathrm{Rad}(C(v, p, q))$.*

In our context $p \neq q$. Assume first that $v \neq p, q$. Observe that if all angles of $\triangle pqr$ are acute (or $\triangle pqr$ is right-angled), then $C(p, q, r)$ is identical to $\circ(p, q, r)$. Otherwise, if one of the angles of $\triangle pqr$ is obtuse, then $C(p, q, r)$ is the circle whose diameter is the longest edge of $\triangle pqr$. If v coincides with either p or q, $C(v, p, q)$ is the circle whose diameter is the segment pq.

Theorem 3 *Let S be a set of n points. The complexity of $V^{(n)}_{\mathcal{K}}(S)$ is $\Omega(n)$.*



(a) Acute triangle (b) Obtuse v (c) Obtuse q

Figure 1: If p, q have a non-empty region in $V^{(n)}_{\mathcal{K}}(S)$, then pq is an edge in $\mathrm{DT}(S)$

Proof. For simplicity assume that each point from S has a unique closest neighbor in S. For each $p \in S$, consider its closest neighbor q. Then, the points of pq lying sufficiently close to p belong to the region of (p, q) in $V^{(n)}_{\mathcal{K}}(S)$, which is thus non-empty. Since no region is thereby encountered more than twice, $V^{(n)}_{\mathcal{K}}(S)$ has at least $\lceil n/2 \rceil$ non-empty regions. □

Theorem 4 *Let S be a set of n points. The complexity of $V^{(n)}_{\mathcal{K}}(S)$ is $O(n^{2+\varepsilon})$ (for any $\varepsilon > 0$).*

Proof. Let a point v belong to a non-empty region of (p, q). No matter if $\triangle vpq$ is acute (Fig. 1(a)), $\triangle vpq$ is obtuse with v the obtuse vertex (Fig. 1(b)), or $\triangle vpq$ is obtuse with p or q the obtuse vertex (Fig. 1(c)), $C(v, p, q)$ cannot contain any other point $x \in S$. Otherwise, regardless of the location of x in $C(v, p, q)$, we will always have $\mathcal{K}(v, (p, q)) > \mathcal{K}(v, (x, q))$, a contradiction. This follows from the fact [6, Lemma 4.14] that given a point set K and its minimum enclosing circle C, removing from K one of the (two or three) points defining C will reduce the radius of the minimum enclosing circle. Thus, there is a circle containing p, q that is empty of any other point of S. Therefore, pq is a Delaunay edge of S. Thus, there are $O(n)$ pairs of sites in S that have non-empty regions in $V^{(n)}_{\mathcal{K}}(S)$. Furthermore, the Voronoi surface of (p, q) is made of a constant number of patches, each of which is "well-behaved" in the sense discussed above. Again, the complexity of the lower envelope of these $O(n)$ surfaces is $O(n^{2+\varepsilon})$ (for any $\varepsilon > 0$). □

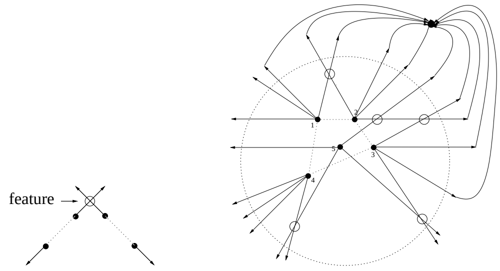
Theorem 5 *Let S be a set of n points. The complexity of $V^{(f)}_{\mathcal{K}}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).*

Proof. Again, we prove the claim via the upper envelope of $\Theta(n^2)$ "well-behaved" Voronoi surfaces. □

4 View Angle

Definition 3 *Given two points p, q, the "view-angle distance" $\mathcal{V}$ from a point v to the unordered pair (p, q) is defined as $\mathcal{V}(v, (p, q)) = \measuredangle pvq$.*

Similarly to $\mathcal{C}$, the view-angle function is undefined at the given points. For fixed points p, q, the curve $\mathcal{V}(v, (p, q)) = \pi$ is the open segment pq, while the curve $\mathcal{V}(v, (p, q)) = 0$ is the line $\overline{pq}$ excluding the closed segment pq. The curve $\mathcal{V}(v, (p, q)) = \pi/2$ is the circle with diameter pq (excluding p and q).



(a) Intersection point (b) Graph

Figure 2: The graph of $V_{\mathcal{V}}^{(n)}(S)$

Theorem 6 *Let S be a set of n points. The complexity of $V_{\mathcal{V}}^{(n)}(S)$ is $\Omega(n^4)$.*

Proof. Let S be a set of n points. Intersections of the complements of two segments defined by two pairs of points (w.r.t. the supporting lines, Fig. 2(a)) are features of $V_{\mathcal{V}}^{(n)}(S)$. Create a geometric graph G on the given points, in which each segment's complement defines two edges. Add one additional point far away from the convex hull of S, and connect it (without intersections) to all the rays (Fig. 2(b)). We can now use the crossing-number lemma for bounding the number of intersections. The lemma tells us that every drawing of a graph with n vertices and $m \geq 4n$ edges (without self or parallel edges) has $\Omega(m^3/n^2)$ crossing points [1, 10]. In our case $m = n(n-1)$, so the number of intersection points in G is $\Omega(n^4)$. $\square$

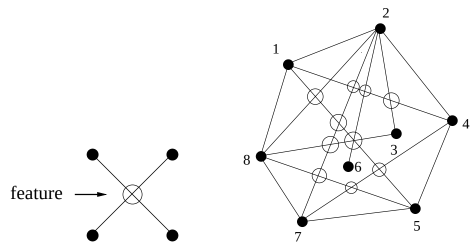
Theorem 7 *Let S be a set of n points. The complexity of both $V_{\mathcal{V}}^{(n|f)}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).*

Proof. For analyzing $V_{\mathcal{V}}^{(n|f)}$(S) consider the function $(-\cos \measuredangle pvq)$ instead of $\measuredangle pvq$. This is allowable since $\cos(\cdot)$ is strictly decreasing in $[0, \pi]$. By the cosine law, $-\cos \measuredangle pvq = (|pq|^2 - |vp|^2 - |vq|^2)/(2|vp||vq|)$. Hence, the $\Theta(n^2)$ Voronoi surfaces fulfill Assumptions 7.1 of [12, p. 188]. As above, apply Theorem 7.7 of [ibid., p. 191] to obtain the claimed bound. $\square$

Theorem 8 *Let S be a set of n points. The complexity of $V_{\mathcal{V}}^{(f)}(S)$ is $\Omega(n^4)$.*

Proof. Given a set S of n points, we count the intersections of pairs of line segments defined by pairs of points of S (Fig. 3(a)). Create a geometric graph on the points of S, in which edges are the line segments connecting pairs of points (Fig. 3(b)). The intersections of segments defined by all pairs of points define features of $V_{\mathcal{V}}^{(f)}(S)$, since along these segments the view-angle function assumes its maximum possible value, π. We can now use the crossing-number lemma for counting these intersections. The graph with n vertices and $m \geq 4n$ edges (without self or parallel edges) has $\Omega(m^3/n^2)$ crossing points [1, 10]. In this case $m = n(n-1)/2$, hence the claimed bound. $\square$

Results by Asano et al. [2] imply that the edges of $V_{\mathcal{V}}^{(n|f)}(S)$ are pieces of polynomial curves of degree at most 3. However, the structure of the part of $V_{\mathcal{V}}^{(f)}(S)$ that lies outside the convex hull $\mathcal{CH}(S)$ of S is fairly simple: it is given by the arrangement of lines supporting the edges of $\mathcal{CH}(S)$. This arrangement can be computed by a standard incremental algorithm in optimal $\Theta(k^2)$ time and space, where k is the number of vertices of $\mathcal{CH}(S)$. Each cell of the arrangement should then be labeled with a pair of sites from S, to the Voronoi region of which it belongs; this extra task can be completed within the same complexity bounds.



(a) Intersection point (b) Graph

Figure 3: The graph of $V_{\mathcal{V}}^{(f)}(S)$

5 Parameterized Perimeter

Definition 4 *Given two points p, q and a real constant $c \geq -1$, the "parameterized perimeter distance" $\mathcal{P}_c$ from a point v to the unordered pair (p, q) is defined as $\mathcal{P}_c(v, (p, q)) = |vp| + |vq| + c \cdot |pq|$.*

We require that $c \geq -1$, since allowing $c < -1$ would result in negative distances. Letting $c = -1$ results in a distance function that equals 0 for all the points on the segment pq. If $c = 0$, we deal with a "sum of distances" distance function introduced in [5] and recently revisited in [13]. For $c = 1$, the above definition yields the "perimeter" distance function $\mathcal{P}(v, (p, q)) = \mathrm{Per}(\triangle vpq)$.

It was proven [8] that the complexity of the 2-site perimeter Voronoi diagram of n points is $O(n^{2+\varepsilon})$. The key observation was that any pair of sites that has a non-empty region in the perimeter diagram also has a non-empty region in the sum-of-distances diagram. Consequently, the number of such pairs is linear in n. Again, one applies the Davenport-Schinzel machinery and conclude the claimed upper bound on the complexity of the diagram. We provide here an alternative and much simpler proof of the same bound, which generalizes to the case of "parameterized perimeter" distance function for any $c \geq 0$.

Theorem 9 *Let S be a set of n points. The complexity of $V_{\mathcal{P}}^{(n)}(S)$ is $O(n^{2+\varepsilon})$ (for any $\varepsilon > 0$).*

Proof. Refer to Fig. 4. Let $p, q \in S$ be two sites which have a non-empty region in $V_{\mathcal{P}}^{(n)}(S)$, and let v be a point in this region. In addition, let ℓ be the perpendicular bisector of the segment pq. Assume, w.l.o.g., that $|vp| \leq |vq|$.

Consider the ellipse O_{vpq} passing through q with v and p as foci. By definition, for any point s inside this

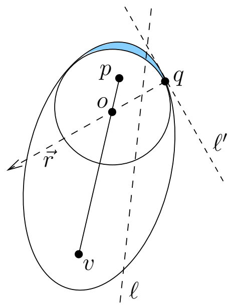


Figure 4: An empty circle containing sites in $\mathcal{P}$

ellipse we have $|vs| + |ps| < |vq| + |pq|$. Therefore,

$$\begin{aligned} \mathcal{P}(v,(p,s)) &= |vs| + |ps| + |vp| \\ &< |vq| + |pq| + |vp| = \mathcal{P}(v,(p,q)). \end{aligned} \tag{1}$$

Hence, $s \notin S$, otherwise v would belong to the region of (p, s) instead of that of (p, q). Therefore, O_{vpq} is empty of any sites other than p and q.

Now consider the line ℓ' that is tangent to O_{vpq} at q, and the ray $\vec{r}$ perpendicular to ℓ' at q and passing through O_{vpq}. It is a known property of ellipses that this ray bisects the angle $\measuredangle vqp$, thus, it intersects the segment vp, say, at point o. The circle C centered at o and passing through q is tangent to O_{vpq} at q (as well as at another point), and is entirely contained in O_{vpq}. Since $|vp| \le |vq|$, it follows that C also contains p. (If p were on the extension of vp in the shaded area, a contradiction would easily be obtained by using the triangle inequality: $|op| > |oq|$, hence $|vp| = |ov| + |op| > |ov| + |oq| > |vq|$, a contradiction to $|vp| \le |vq|$.) Since O_{vpq} is empty of sites (except p, q), so is C. Therefore, pq is an edge of the Delaunay triangulation of S. The number of such edges is linear in n. Hence, there are $\Theta(n)$ respective surfaces of these pairs of sites. One can now apply the standard Davenport-Schinzel machinery, and the claim follows. $\square$

Finally, we state the following theorem.

Theorem 10 *Let S be a set of n points.*
(a) The complexity of $V^{(n)}_{\mathcal{P}_{-1}}(S)$ is $\Omega(n^4)$ and $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).
(b) If there is a unique closest pair $p, q \in S$, then when $c \to \infty$, the complexity of $V^{(n)}_{\mathcal{P}_c}(S)$ is asymptotically 1.
(c) For $c \ge 0$, the complexity of $V^{(n)}_{\mathcal{P}_c}(S)$ is $O(n^{2+\varepsilon})$ (for any $\varepsilon > 0$).

The easy proofs of Theorems 10(a,b), as well as the proof of Theorem 10(c) (which is a nontrivial generalization of the proof of the special case $c = 1$), are provided in the full version of the paper.

6 Conclusion

In this paper, we have investigated 2-site Voronoi diagrams of point sets with respect to a few geometric distance functions. The Voronoi structures obtained in this way cannot be explained in terms of the previously known kinds of Voronoi diagrams (which is the case for the 2-site distance functions thoroughly analyzed in [5]), what makes them particularly interesting. On the other hand, our results can be exploited to advance research on Voronoi diagram for segments. Potential directions for future work include consideration of other distance functions, and generalizations to higher dimensions and to k-site Voronoi diagrams.

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