Coloring Planar Homothets and Three-Dimensional Hypergraphs

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Abstract

We prove that every set of homothetic copies of a given convex body in the plane can be colored with four colors so that any point covered by at least two copies is covered by two copies with distinct colors. This generalizes a previous result from Smorodinsky [8]. We also show a relation between our proof and Schnyder's characterization of planar graphs. Using this characterization we generalize the first result and show that $k \geq 2$, every three-dimensional hypergraph can be colored with $6(k-1)$ colors so that every hyperedge e contains $\min\{|e|, k\}$ vertices with mutually distinct colors. Furthermore, we also show that at least $2k$ colors might be necessary. This refines a previous result from Aloupis et al. [1].

1 Introduction

The well-known graph coloring problem has several natural generalizations to set systems, or hypergraphs. A proper coloring of a hypergraph can be defined such that no hyperedge is monochromatic, or such that every hyperedge contains some minimum number of distinct colors, for instance. A rich literature exists on these topics; in particular, the two-colorability of hypergraphs (also known as property B), has been well-studied since the sixties [4].

In this paper, we concentrate on coloring geometric hypergraphs, defined by simple objects in the plane. Those hypergraphs serve as models for wireless sensor networks, and associated coloring problems have been investigated recently. Smorodinsky [8] investigated the chromatic number of such geometric hypergraphs, defined as the minimum number of colors required to make every hyperedge non-monochromatic. He considered hypergraphs induced by a collection S of regions in the plane, whose vertex set is S, and the hyperedges are all subsets $S' \subseteq S$ for which there exists a point p such that $S' = \{R \in S : p \in R\}$ (i.e., the regions that contain p). He proved the following result.

Theorem 1
- *Any hypergraph that is induced by a family of n simple Jordan regions such that the union complexity of any m of them is given by $u(m)$ and $u(m)/m$ is non-decreasing is $O(u(n)/n)$-colorable. In particular, any finite family of pseudodisks can be colored with a constant number of colors.*
- *Any hypergraph induced by a finite family of disks is four-colorable*

Later, Aloupis, *et. al.* [1] considered parameterized chromatic numbers. In particular, they studied the quantity $c(k)$, defined as the minimum number of colors required to color a given hypergraph, such that every (sufficiently large) hyperedge has at least k vertices with k distinct colors. This generalizes the previous notion of chromatic number, which corresponds to the case $k = 2$. They proved the following.

Theorem 2 *Any finite family of pseudo-disks in the plane can be colored with $O(k)$ colors in such a way that any point covered by r pseudo-disks is covered by $\min\{r, k\}$ pseudo-disks with distinct colors. For the special case of disks, the number of colors is at most $24k + 1$.*

Our results First, we show in section 3 that Smorodinsky's result for disks holds for every convex body. The proof is similar to that of Smorodinsky, but the graph is constructed in a different way, reminiscent from Schnyder's characterization of planar graphs. This characterization is closely related to the concept of *dimension* of graphs and hypergraphs. Thus, in section 4 we will show the connection between both results and study the chromatic number for three-dimensional hypergraphs. Among other results, we will show that this number is at most $6(k-1)$. This improves the constant of Theorem 2 for this class of hypergraphs, which includes in particular hypergraphs induced by homothets of a triangle. In section 5, we will show a lower bound for all the above problems.

Due to length constraints, some of the proofs have been omitted and/or simplified. A longer version of this paper can be found in [2].

Definitions and notations We consider hypergraphs defined by *ranges*, which are open convex bodies of the form $Q \subset \mathbb{R}^2$ containing the origin. The *scaling* of Q by a factor $\lambda \in \mathbb{R}^+$ is the set $\{\lambda x : x \in Q\}$. The *translate* of Q by a vector $t \in \mathbb{R}^2$ is the set $\{x + t : x \in Q\}$. The *homothet* of Q of *center* t and *scaling* λ is the set $\{\lambda x + t : x \in Q\}$.

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Given a collection S of points in the plane, the *primal hypergraph* defined by these points and a range Q has S as vertex set, and $\{S \cap Q' : Q' \text{ homothet of } Q\}$ as hyperedge set. Similarly, the *dual hypergraph* defined by a set S of homothets of Q has S as vertex set, and the hyperedges are all subsets $S' \subseteq S$ for which there exists a point $p \in \mathbb{R}^2$ such that $S' = \{R \in S : p \in R\}$ (i.e., the set of ranges that contain p).

For a given range Q, the chromatic number $c_Q(k)$ is the minimum number c such that every primal hypergraph (induced by a set of points) can be colored with c colors, so that every hyperedge of size r contains $\min\{r, k\}$ vertices with mutually distinct colors. Similarly, the chromatic number $\bar{c}_Q(k)$ is the smallest number c such that every dual hypergraph (induced by a set of homothets of Q) can be c-colored so that every hyperedge of size r contains $\min\{r, k\}$ vertices with mutually distinct colors.

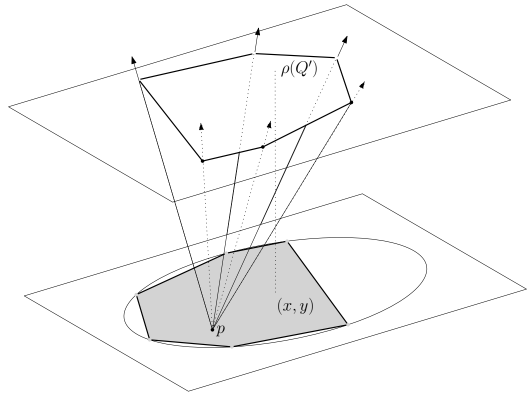

Figure 1: Mapping of a range Q' (in grey) to a point in $\rho(Q') \in \mathbb{R}^3$ and mapping of a point p to a cone. The main property of the mapping is that inclusions are reversed (i.e., a point p is inside Q' if and only if the cone $\pi(p)$ contains $\rho(Q')$).

2 Primal Problem

As a warm-up, we consider the primal version of the problem for $k = 2$. We are interested in giving a coloring of the points of S such that any homothet of Q that contains two or more points of S contains two points of different colors.

Given a set of points S and range Q, the *generalized Delaunay graph* of S induced by Q is a graph $G = (S, E)$ with S as vertex set. For any two points $p, q \in S$, their edge pq is in E if and only there exists a homothet Q' of Q such that Q' contains p, q and no other point of S (in other words $pq \in E \Leftrightarrow Q' \cap S = \{p, q\}$). Note that the Delaunay graph induced by disks corresponds to the classic Delaunay triangulation.

Lemma 3 *[6] For any convex range Q and set of points S, the Delaunay graph of S induced by Q is planar*

The above result has been rediscovered many times along the literature for different types of ranges. See the extended version of this paper [2] for more details.

Theorem 4 *For any convex range Q we have $c_Q(2) \leq 4$.*

Proof. Consider the Delaunay graph of S induced by Q. By Lemma 3, this graph is planar and thus can be four colored. In the following we will show that this coloring is also a valid coloring for our purpose. Let Q' be any homothet of Q containing two or more points of S. Note that if Q' contains exactly two points p and q we have $pq \in E$ by definition of generalized Delaunay graph. In particular, the colors of p and q are different, hence Q' cannot be monochromatic. If Q' has strictly more than two points of S, we continuously shrink it until it contains exactly two points p, q of S (more details of this process can be seen in [2]). As before we have that both $p, q \in Q'$ and $pq \in E$, hence Q' cannot be monochromatic. $\square$

3 Dual Problem

We now consider the dual version of the problem. That is, we are given a set S of homothets of Q. We say a point is *k-deep* whenever it is covered by at least k elements of S. We apply the same technique as in the primal case: given S, we construct a planar graph $G(S) = (S, E(S))$. The main property of G is that any two-deep point will be covered by two adjacent vertices of $G(S)$.

Let Q' be a homothet of Q with center (x, y) and scaling d. We denote by $\rho(Q')$ the point $(x, y, d) \in \mathbb{R}^3$. Given a set S of homothets of Q, we define $\rho(S) = \{\rho(Q') : Q' \in S\}$.

Similarly, we associate with every point $p = (x, y, d) \in \mathbb{R}^3$ the *cone* $\pi(p)$ defined as follows. Let Q^* be the reflexion of Q about its center. The intersection of $\pi(p)$ with the horizontal plane of height $z \geq d$ is the homothet of Q^* with center (x, y) and scaling $z - d$. The intersection of $\pi(p)$ with a horizontal plane of height $z < d$ is empty. Note that the cone $\pi(p)$ so defined is convex.

We now proceed to define the graph $G(S)$. Its vertex set is S, and two elements Q', Q'' of S are adjacent if and only if there exists a point $p \in \mathbb{R}^3$ such that $\pi(p) \cap \rho(S) = \{\rho(Q'), \rho(Q'')\}$. Thus $G(S)$ is a Delaunay graph in $\mathbb{R}^3$, with cones $\pi(p)$ as ranges.

For any point $p \in \mathbb{R}^2$, let S_p be the set of ranges containing p (i.e., $S_p = \{Q' \in S : p \in Q'\}$). As in the primal case, a valid coloring of G will suffice for our problem:

Lemma 5 *For any $p \in \mathbb{R}^2$ such that $|S_p| \geq 2$, there exist $Q', Q'' \in S_p$ such that $Q'Q'' \in E(S)$.*

Proof. For every point $p = (x, y) \in \mathbb{R}^2$ we consider its cone $\pi((x, y, 0))$. The number of points of $\rho(S)$ contained in this cone is the number of elements of S containing p. We translate vertically upwards $\pi((x, y, 0))$ until it contains exactly two points $\rho(Q')$ and $\rho(Q'')$ (more details of this process can be seen in [2]). These two form an edge in $G(S)$. Since, the translated cone is contained in $\pi(p)$ both Q' and Q'' belong to S_p. ☐

Lemma 6 *The graph $G(S)$ is planar.*

Proof. By definition of $E(S)$, we know that for every edge $Q'Q'' \in E$ there exists $p \in \mathbb{R}^3$ such that $\pi(p) \cap \rho(S) = \{\rho(Q'), \rho(Q'')\}$. We draw the edge QQ'' as the projection on a horizontal plane of the two line segments connecting respectively $\rho(Q')$ and $\rho(Q'')$ with p.

Note that crossings involving two edges with a common endpoint can be eliminated, so we can simply show that the proposed embedding has no crossing involving vertex-disjoint edges. Consider two such edges uu' and vv', and their corresponding witness cones $\pi_1 \ni u, u'$ and $\pi_2 \ni v, v'$. We must have $u \notin \pi_2$ and $v \notin \pi_1$.

Suppose that the projections of the segments connecting u with the apex of π_1 and v with the apex of π_2 cross at an interior point x. Consider the vertical line ℓ that passes through x: by construction, this line must intersect with both segments at points a and b, respectively. Without loss of generality we assume that a has lower z coordinate than b. From the convexity of π_1, we have $a \in \pi_1$. However, this yields a contradiction, since $v \subseteq \pi(b) \subseteq \pi(a) \subseteq \pi_1$ and we assumed $v \notin \pi_1$. ☐

Theorem 7 *For any convex range Q we have $\bar{c}_Q(2) \leq 4$.*

4 Coloring Three Dimensional Hypergraphs

The proof of Lemma 6 actually generalizes the "easy" direction of Schnyder's characterization of planar graphs. We first give a brief overview of this fundamental result.

4.1 Poset dimension and Schnyder's theorem

The *vertex-edge incidence poset* of a graph $G = (V, E)$ is a bipartite poset $P = (V \cup E, \preceq_P)$, such that $e \preceq_P v$ if and only if $e \in E$, $v \in V$, and $v \in e$. The *dimension* of a poset $P = (S, \preceq_P)$ is the smallest d such that there exists a injective mapping $f : S \to \mathbb{R}^d$, such that $u \preceq_P v$ if and only if $f(u) \leq f(v)$, where the order $\leq$ is the componentwise partial order on d-dimensional vectors.

Theorem 8 ([7]) *A graph is planar if and only if its vertex-edge incidence poset has dimension at most three.*

The *easy* direction of Schnyder's theorem consists of showing that every graph with vertex-edge incidence poset of dimension at most three is planar. The non-crossing drawing that is considered in the proof is similar to ours, and simply consists, for every edge $e = uv$, of projecting the two line segments $f(e)f(u)$, and $f(e)f(v)$ onto the plane $x + y + z = 0$. This can be identified as a special case of our proof, in which Q is an (equilateral) triangle.

In fact, Lemma 5 directly yields the following corollary.

Corollary 9 *Every hypergraph with vertex-edge incidence poset of dimension at most three is four-colorable.*

4.2 Polychromatic coloring of three-dimensional hypergraphs

We now adapt the above corollary for higher values of k. That is, we are given a three-dimensional hypergraph $G = (V, H)$ and a constant $k \geq 2$. We would like to color the vertices of G such that any hyperedge $e \in H$ contains at least $\min\{|e|, k\}$ vertices with different colors. We will denote by $c_3(k)$ to the minimum number of colors necessary so that any three dimensional hypergraph can be colored. We note that the problem is self-dual: any instance of the dual problem can be transformed into a primal coloring problem by symmetry with respect to the point $(1, 1, 1)$ (assuming that all points are mapped to the interior of the unit cube). Hence, any result for the primal coloring problem will apply to the dual and vice-versa.

In order to avoid degeneracies we assume that no two vertices of G in the mapping share an x, y or z coordinate. For any hyperededge $e \in H$, we define the x-extreme of e as the point $x(e) \in e$ whose mapping has smallest x-coordinate. Analogously we define the y and z-extremes and denote them $y(e)$ and $z(e)$, respectively. For any hyperedgeedge $e \in H$, there exist many points in $\mathbb{R}^3$ that dominate the points of e. We will assume that e is mapped to the point $q_e \in \mathbb{R}^3$ whose x coordinate is equal to the x coordinate of $x(e)$ (analogously for the y and z coordinates). We say that a hyperedge e is *degenerate* if two extremes of e are equal.

Lemma 10 *For any $k \geq 3$, the graph G has at most $3n$ degenerate hyperedges of size exactly k*

For any $2 \leq k \leq n$, we define the graph $G_k(S) = (S, E_k)$, where for any $u, v \in S$ we have $uv \in E_k$ if and only if there exists a point a point $q \in \mathbb{R}^3$ that dominates u, v and at most $k - 2$ other points of S

(that is, we replace hyperedges of size k or less by cliques). The main property of this graph is that any coloring of $G_k(S)$ (in the classic graph coloring sense) induces a polychromatic coloring of G.

We now bound the total number of edges of $G_k(S)$:

Lemma 11 *For any set S of points and $2 \leq k \leq n$, graph $G_k(S)$ has at most $3(k-1)n - 6$ edges*

The bound on the number combined with the minimum degree coloring technique [3] allows us to obtain a coloring of G:

Theorem 12 *For any $k \geq 2$, and three-dimensional hypergraph $G = (V, H)$, the vertices of G can be colored with $6(k-1)$ colors so that any hyperedge $e \in H$ contains $\min\{|e|, k\}$ points with distinct colors. In other words $c_3(k) \leq 6k - 6$*

4.3 Coloring triangles

In this section we will give a very simple application of the previous result. It is easy to show that triangle containment posets (that is, posets of inclusion of homothets of a given triangle) have dimension at most 3. Thus the dual hypergraphs induced by collections of triangles have dimension at most 3, and our result applies.

Theorem 13 *Triangle containment posets have dimension at most 3.*

Corollary 14 *For any $k \geq 3$, any set S of homothets of a triangle can be colored with $6(k-1)$ colors so that any point $p \in \mathbb{R}^2$ covered by r homothets is covered by $\min\{r, k\}$ homothets with distinct colors.*

Note that this result extends the result of Theorem 7 (for the case in which Q is a triangle) to larger values of k. Other than being more general, this proof shows some interesting properties.

Consider the primal variant of Corollary 14: we would like to show that a set S of points can be colored with few colors such that any homothet Δ of a fixed triangle will contain $\min\{|\Delta \cap S|, k\}$ points with different colors. Although the problems are clearly similar, it is not easy to see that they are equivalent. However, by Theorem 13, we know that any instance of the dual problem will generate a three-dimensional hypergraph. Since the dual of a three dimensional hypergraph is another three dimensional hypergraph (as mentioned in Section 4.2), we can apply Theorem 12 to both problems.

In the following Section we will show lower bounds for $c_Q(k)$ and $\bar{c}_Q(k)$ for many different ranges (among them the triangle). Since triangle containment posets have dimension at most 3, this will directly give the same lower bounds for $c_3(k)$. That is, $c_3(k) \geq c_\Delta(k)$, where Δ is any triangular range.

5 Lower Bound

In this section we will give a lower bound for $c_Q(k)$. For that we will use the well known concept of normal direction of Q in a point p (that is, the normal of Q at the boundary point p is the unit vector that is orthogonal to the halfplane that passes through p and supports Q, pointing outwards from Q). We say that a range has m distinct normal directions if there exist m different points such that for any two points, their normals are linearly independent. Note that any affine transformation of a square has two normal directions, a triangle three and a circle has infinitely many.

Lemma 15 *Any range Q with at least three distinct normal directions satisfies $c_Q(k) \geq 4\lfloor k/2 \rfloor$ and $\bar{c}_Q(k) \geq 4\lfloor k/2 \rfloor$.*

The Lemma shows that the upper bounds of Sections 2 and 3 are tight for any range with at three distinct normal directions. Notice that the only bounded shape that does not have three distinct normal directions is the square (and any affine transformation). The above reasonings can be adapted for this case, but for a weaker $c_Q(k) \geq 3\lfloor k/2 \rfloor$ lower bound.

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References

[1] G. Aloupis, J. Cardinal, S. Collette, S. Langerman, and S. Smorodinsky. Coloring geometric range spaces. *Discrete & Computational Geometry*, 41(2):348–362, 2009.

[2] J. Cardinal and M. Korman. Coloring planar homothets and three-dimensional hypergraphs. *CoRR*, abs/1101.0565, 2010.

[3] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, third edition, 2005.

[4] P. Erdős. On a combinatorial problem. *Nordisk Mat. Tidskr.*, 11:5–10, 1963.

[5] E. Horev, R. Krakovski, and S. Smorodinsky. Conflict-free coloring made stronger. In *SWAT*, pages 105–117, 2010.

[6] D. Sarioz. Generalized delaunay graphs with respect to any convex set are plane graphs. *CoRR*, abs/1012.4881, 2010.

[7] W. Schnyder. Planar graphs and poset dimension. *Order*, 5:323–343, 1989.

[8] S. Smorodinsky. On the chromatic number of some geometric hypergraphs. *SIAM Journal on Discrete Mathematics*, 21(3):676–687, 2007.