Design of Antigravity Slopes for Visual Illusion

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Abstract

This paper presents a method for designing solid shapes containing slopes where the orientation appears opposite to the actual orientation when observed from a unique vantage point. The resulting solids generate a new type of visual illusion, which we call "impossible motion", in which balls placed on the slopes appear to roll uphill, thereby defying the law of gravity. This is possible, because a single retinal image lacks depth information and human visual perception tries to interpret images as the most familiar shape, even though there are infinitely many possible interpretations. We specify the set of all possible solids represented by a single picture as the set of a system of equations and inequalities, and then relax the constraints in such a way that the antigravity slopes can be reconstructed. We present this design procedure with examples.

1 Introduction

This paper presents a computational approach to the design of a new visual illusion. Visual illusion is a perceptual behavior where what we "see" differs from the physical reality. This phenomenon is important in vision science because it helps us to understand the basis of human perception [5, 6]. Numerous traditional visual illusions are known, most of which are generated by two-dimensional pictures and their associated spatial manipulations [4, 10].

However, very few visual illusions are known that make use of three-dimensional solid shapes. An early example was the Ames room, where a person appears to become taller when moving from one corner of the room to another [3]. Other examples include impossible solids produced using a hidden-gap trick [2], and others without hidden gaps [8]. This latter classification was extended to include a new type of illusion called "impossible motion" [9].

Design of illusions using solids requires the application of mathematics, because this process can be counterintuitive.

This paper concentrates on one class of such solids called "antigravity slopes", in which balls appear to roll uphill against the law of gravity and produce the appearance of "impossible motion".

In Section 2, we specify the set of all possible solids represented by a picture, and in Section 3 we show that antigravity slopes cannot be constructed using that formulation. In Section 4, we remove some of the constraints, so design of antigravity slopes becomes possible. We show some examples in Section 5, and provide concluding remarks in Section 6.

2 Reconstruction of a Solid from a Picture

In this paper we consider polyhedrons, which are solids bounded by planar faces. We do not consider solids with curved surfaces. As shown in Fig. 1, we assume that a viewpoint is fixed at the origin of an xyz coordinate system, and that a picture of a solid is fixed at the plane $z = 1$.

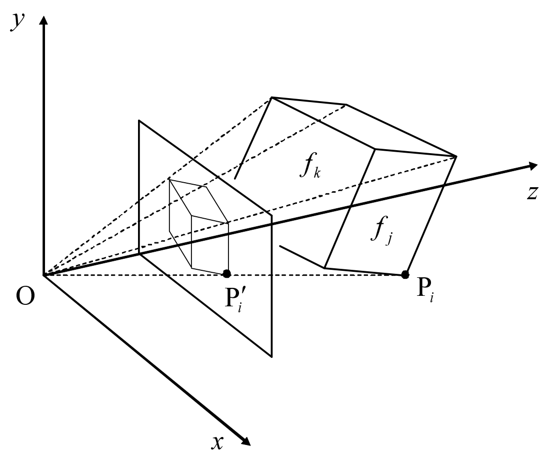

Figure 1: Central projection of a solid.

Suppose that we are provided with the relative relations among the vertices and the faces of the solid. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $F = \{f_1, f_2, \ldots, f_m\}$ be the set of vertices and that of faces of the solid, respectively. Let ON(v_i, f_j) represent a predicate stating that the vertex v_i is on the face f_j. Similarly, let NEARER(v_i, f_j) indicate that "v_i is nearer to the viewpoint than the plane containing f_j", while FARTHER(v_i, f_j) indicates that "v_i is farther away than the plane containing f_j".

We assume that, in addition to the picture, we are given all predicates satisfied by the solid shape and

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that we are interested in determining the reconstructability of a solid from the picture.

Let $(x_i, y_i, 1)$ be the coordinates of the i-th vertex on the picture plane. The original vertex on the solid should be on the half-line emanating from the origin that passes through the associated vertex on the picture plane, so we can express the coordinates of the original vertex in the space as $(x_i/t_i, y_i/t_i, 1/t_i)$, where t_i is an unknown parameter representing the inverse of the depth of the vertex from the viewpoint measured along the z axis.

Let

$$a_j x + b_j y + c_j z + 1 = 0 \quad (1)$$

be the plane containing the j-th face, where a_j, b_j and c_j are unknown.

Suppose that $\mathrm{ON}(v_i, f_j)$ is true. We can substitute the coordinates of v_i into the equation of f_j, and we find

$$a_j x_i + b_j y_i + c_j + t_i = 0, \quad (2)$$

which is linear for the unknowns. Similarly, if $\mathrm{NEARER}(v_i, f_j)$ is true, we find

$$a_j x_i + b_j y_i + c_j + t_i < 0, \quad (3)$$

and if $\mathrm{FARTHER}(v_i, f_j)$ is true, we find

$$a_j x_i + b_j y_i + c_j + t_i > 0. \quad (4)$$

We combine all equations of the form (2) for each ON predicate, and denote the resulting system of equations as

$$Aw = 0, \quad (5)$$

where $w = (z_1, \ldots, z_n, a_1, b_1, c_1, \ldots, a_m, b_m, c_m)^{\mathrm{t}}$ is the vector of unknown variables (t represents the transpose) and A is a constant matrix. Similarly, we collect all inequalities of the forms (3) and (4), and denote the resulting system of inequalities as

$$Bw > 0, \quad (6)$$

where B is a constant matrix, and the inequality symbol ">" represents componentwise inequalities, each of which is either ">" or "<".

We can prove that the picture represents a three-dimensional solid if and only if the system consisting of (5) and (6) has solutions [7]. Furthermore, the system of (5) and (6) is sensitive to errors of vertex positions in the picture, but a robust method for reconstructing the solid from the picture was also established [7].

3 Impossibility of Antigravity Slopes

We concentrate on antigravity slopes as a typical class of impossible motions. Let us consider the picture of a simple solid shown in Fig. 2, in which a slope is supported by two columns standing on a base plate.

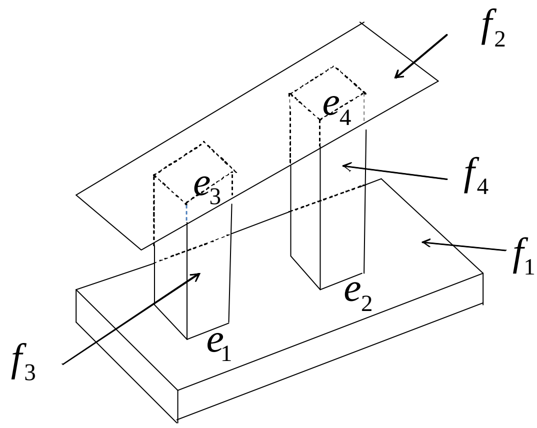


Figure 2: Slope supported by two columns.

The broken lines represent the hidden parts. However, to avoid unnecessary complexity, some hidden parts are not shown.

Let f_1 denote the top face of the base plate, and f_2 denote the slope plane. For each of the two columns, we assume that all four lower vertices are on f_1 and are farther away than f_2, while all four upper vertices are on f_2 and are nearer than f_1. We also assume that f_1 is horizontal. Thus, we expect that the slope f_2 tilts to the left, that is, the right end of f_2 is higher than the left end. Indeed, the system of equations (5) and inequalities (6) accepts such a slope as its solution.

Now we ask whether the set of the solutions contains a slope that tilts to the right. The answer is "no". In every solid whose projection matches that of the picture shown in Fig. 2, the slope f_2 tilts to the left. This can be understood in the following way. As shown in Fig. 2, let f_3 and f_4 be the right front faces of the left and right columns, e_1 and e_2 the lower edges of f_3 and f_4, and e_3 and e_4 the upper edges of f_3 and f_4, respectively. Because the edges e_1 and e_2 are collinear in the picture plane, and they are on f_1, then they must also be collinear in three-dimensional space. Let l_1 be a line in the space containing e_1 and e_2. e_3 and e_4 are collinear in the picture plane and they are on f_2, so they are collinear in three-dimensional space. Let l_2 be the line containing e_3 and e_4. Note that e_1 and e_3 are coplanar because they are on f_3; thus, l_1 and l_2 are coplanar, which implies that f_3 and f_4 are coplanar. l_1 and l_2 meet to the left of the solid, so the slope f_1 tilts to the left.

This property holds for any solution of the system of (5) and (6) associated with the picture shown in Fig. 2. Therefore, it is impossible to construct a slope that tilts to the right from the picture shown in Fig. 2.

4 Construction of an Antigravity Slope

Our goal is to construct a slope that tilts to the right, even though such a slope is not contained in the solutions of (5) and (6). Thus, we must modify the picture in such a way that the visible part of the solid does not change in the picture plane and the solutions of (5) and (6) contain a slope tilting to the right. To achieve this, we can modify the picture around the upper parts of the two columns because they are hidden by the slope. The vertices at the top of the columns can be moved slightly along the associated vertical edges of the columns. Here "slightly" means that the movements of the vertices are restricted to the area covered by f_2 in the picture plane.

Let v_i be one of the four top vertices of the left or right column, and let e_j be the vertical edge of the column incident to v_i. Let $(\alpha_j, \beta_j, 0)$ be the unit vector parallel to e_j in the picture plane. We replace the coordinates $(x_i, y_i, 1)$ of the vertex v_i with

$$(x_i + s_i\alpha_j, y_i + s_i\beta_j, 1), \tag{7}$$

where s_i is a new unknown parameter. Then, instead of the equation (2), the predicate $\mathrm{ON}(v_i, f_j)$ is represented by

$$a_j(x_i + s_i\alpha_j) + b_j(y_i + s_i\beta_j) + c_j + t_i = 0. \tag{8}$$

The inequalities of the forms (3) and (4) are also modified by replacing x_i and y_i with $x_i + s_i\alpha_j$ and $y_i + s_i\beta_j$, respectively. We change the equations and inequalities associated with all the upper vertices of the columns, and denote the resulting equations and inequalities as

$$\bar{A}(s)w = 0, \tag{9}$$
$$\bar{B}(s)w > 0, \tag{10}$$

where $s = (s_1, s_2, \ldots, s_k)$ is the vector of unknown parameters introduced by the movement of the hidden vertices (k denotes the number of hidden vertices to be moved), and $\bar{A}(s)$ and $\bar{B}(s)$ are the resulting coefficient matrices corresponding to the equations (5) and the inequalities (6).

The new system of equations (9) and (10) allows a solution corresponding to a slope tilting to the right, that is, an antigravity slope.

However, (9) and (10) are nonlinear because the matrices $\bar{A}(s)$ and $\bar{B}(s)$ contain unknown variables. Thus, unlike the system of (5) and (6), it is not straightforward to specify the set of all solutions. To overcome this difficulty, we employ the following convention instead of solving (9) and (10) directly.

Our solution is explained using the example shown in Fig. 2. Let v_1, v_2, v_3, v_4 be the top vertices of the left column, and v_5, v_6, v_7, v_8 be the top vertices of the right column. Thus, we have

$$\mathrm{ON}(v_i, f_2), \quad i = 1, 2, \ldots, 8 \tag{11}$$

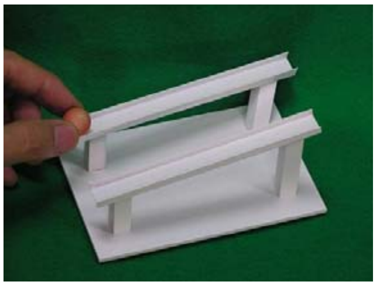
from the original solid structure. From these we accept two predicates,

$$\mathrm{ON}(v_1, f_2) \quad \text{and} \quad \mathrm{ON}(v_5, f_2), \tag{12}$$

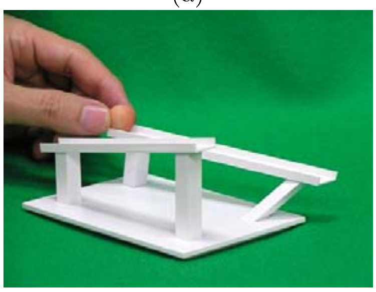
but ignore the other six, and reconstruct the equations (5) and the inequalities (6). As this system produces solutions in which slopes tilt to the right, we choose one of these slopes. In this solid, the six vertices $v_2, v_3, v_4, v_6, v_7, v_8$ are not necessarily on f_2, because the associated equations were ignored. We move these vertices in the three-dimensional space along their associated edges such that they are on f_2, until we obtain a solid in which all the original incidence relations are satisfied. In this solid, some of the vertices move from their original positions, but they are hidden in the slope and movements are restricted on the extended part of the visible edges. Thus, the visible part of the solid remains the same as that shown in the original picture. This describes our method for constructing antigravity slopes.

5 Examples

Figs 3, 4 and 5 show examples of antigravity slopes constructed by this method. In each figure, (a) shows a solid that looks the same as that represented by the original picture, but the orientations of the slopes are perceived as opposite to the actual orientation, and (b) shows the same solid seen from another angle.



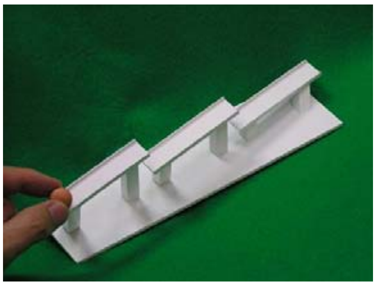
(a)



(b)

Figure 3: "Antigravity parallel slopes".

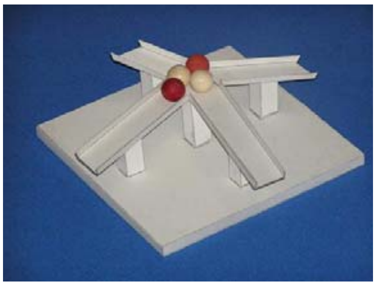
These examples generate impossible motions in that when we place balls on the slopes they appear to roll uphill against the slope, thereby defying the law of gravity. The impossible motion generated by



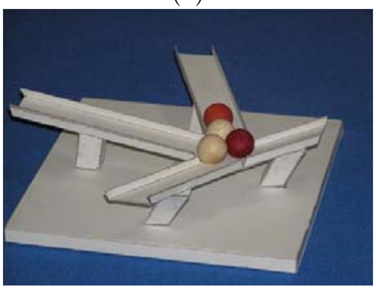
(a)

(b)

Figure 4: "Antigravity cascade of three slopes".



(a)



(b)

Figure 5: "Magnet-like slopes."

the solid shown in Fig. 5 was awarded first prize in the 2010 Best Illusion of the Year Contest held in Florida in May 2010 [1].

6 Concluding Remarks

We described our basic method for constructing antigravity slopes. When we observe antigravity slopes from a specific viewpoint, the orientations of the slopes appear opposite to the actual orientations, which generates a visual illusion of the impossible motion of rolling balls. This is a new computational approach to producing visual illusions. Future tasks include increasing the number of antigravity slope variants, extension to other types of impossible motion and the extension of the method to solids with curved faces. We intend to study human visual perception through visual illusions of impossible motion. We will address these basic research questions in future studies.

We also aim to increase possible applications for antigravity slopes, by developing methods for decreasing the strength of the illusion. It is known that one of the reasons for natural congestion of traffic flows on highways is driver misperception of slope orientation. If we can understand the mechanisms of human illusion in slope perception, we might inform the shape of new highways in which the true orientation of slopes can be easily perceived. We might also suggest possible methods for manipulating the environment around existing highways such that the illusion of slope is prevented.

Acknowledgments

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